THE EFFECTIVE POTENTIAL FOR THE COLEMAN-WEINBERG MODEL

by

ROSS TAYLOR BATES

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We accept this thesis as conforming
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Nathan Weiss
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Ross T. Bates

Department of Physics

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

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Abstract

Gauge theories which have a phase transition could be useful in the study of quark confinement. One of the simplest theories containing a phase transition is the Coleman-Weinberg model of massless scalar electrodynamics. The calculation of the renormalized effective potential for the Coleman-Weinberg model is reviewed in detail using the path integral formalism. The effective potential is evaluated at the one-loop level to show that the model exhibits dynamical symmetry breaking at zero temperature. The divergent parts are shown to be renormalizable to two-loop order. The temperature dependence of the effective potential is then calculated to one-loop in order to demonstrate that the symmetry of the model is restored at high temperature, indicating a phase transition. Finally, for models which exhibit this type of behaviour, applications to SU(n) theories of quarks are discussed.

Table of Contents

List of Figures

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I. INTRODUCTION

Recently in particle physics there has been much interest in SU(n) gauge theories which describe quarks and their interactions. Individual quarks have not been observed and hence it is believed that they exist only in bound states. However, at sufficiently high temperatures it is thought that these bound states could possibly undergo a phase transition into an unconfined phase of free quarks. One can describe the confined nature of the quarks at zero temperature in terms of a symmetry in the underlying gauge theory. This symmetry will not be present in the high temperature unconfined phase. (When the state of a system does not possess the same symmetry as the underlying theory describing it, the symmetry is said to have been spontaneously broken. A detailed review of symmetry breaking {2} can be found in Abers and Lee.) Thus if we wish to include both phases in our theory, we must construct a model containing an intact symmetry at zero temperature which can be broken at high temperature.

With the eventual construction of such a model as the motivation, we need to develop some calculational techniques. In particular we will look at the effective potential method. The effective potential describes the ground state of the system, and from it many physical results can be derived. The effective potential method is a semi-classical approximation in that if one calculates it perturbatively, the lowest order term is the classical solution with the higher order terms being the quantum corrections. We illustrate the techniques involved through an example by doing a detailed calculation of the effective potential for the Coleman-Weinberg model {6}. This calculation represents a review of work done in the literature by several authors {6,9,13,14} on various aspects of the model, and constitutes the main part of this thesis.

The Coleman-Weinberg model describes a photon field which is minimally coupled to a charged, massless, self-interacting scalar field. It was chosen because it possesses the two phase property we desire, and yet is simple enough to be used for an example. At zero temperature the symmetry of the model is broken dynamically by electromagnetic radiative corrections to the lowest order (classical) approximation. Note however that the techniques developed will be equally applicable to models where the symmetry is broken explicitly in the lowest order. The temperature dependence of the model is such that the broken symmetry is restored in the high temperature limit. Although there are some differences, this is the same behaviour we need for the theory describing quarks, and the techniques developed for the Coleman-Weinberg model should be applicable for both.

An outline of the thesis is presented below.

Chapter Two summarizes some of the formalism which will be needed, and lists the main results. For the reader unfamiliar with the subject matter, references are supplied to the appropriate area in the literature where details may be found.

The path integral formulation of the theory and the effective potential are introduced, and a general outline of the procedure used to calculate the effective potential is given. Also included is a brief discussion of the dimensional regularization techniques used in evaluating Feynman integrals.

Chapter Three contains a detailed calculation of the effective potential for the Coleman-Weinberg model at zero temperature. It is evaluated and renormalized at the one-loop level in a general Lorentz gauge. The equivalence of the conventional and minimal subtraction renormalization schemes is demonstrated. The broken symmetry of the theory is also shown. Then the renormalized two-loop effective potential is calculated in Landau gauge to show the explicit cancellation of the divergent parts. Having illustrated the method of obtaining them, the remaining finite terms are left in integral form and will not be evaluated here. Finally the gauge dependence of the effective potential is discussed, as well as its effect on the physical results of the model.

Chapter Four then studies the temperature dependence of the Coleman-Weinberg model at the one-loop level. The changes in procedure from the zero temperature case are mentioned, and the temperature dependence of the effective potential calculated. The result is in integral form and must be evaluated by approximation techniques. A high temperature expansion is performed which shows that the symmetry of the model is restored at high temperature, indicating a phase transition.

Chapter Five summarizes the techniques developed in the preceding chapters. It then gives a discussion of their application to other gauge theories where new research is possible, which was the original motivation behind reviewing the Coleman-Weinberg calculation.

II. FORMALISM

This chapter will briefly review some of the basic definitions and procedures needed as background to the developments in subsequent chapters. Only the main results will be given, with the reader referred to the literature for more detailed discussion. It will be assumed that the reader is at least partially familiar with perturbation theory, Green's functions, Feynman diagrams and functional methods. These topics may all be found described in various references {4,5,12,16}.

2.1 The Path Integral

A basic tool we will need is the fundamental path integral over functional space. Consider a field theory with degrees of freedom $Q_1(x), Q_2(x), \ldots.$ which is described by an action $S(Q)$, where the action is quadratic in the fields $Q_\alpha(x)$. For this theory consider the generating functional $Z(J)$ in the prescence of a source $J(x)$. The path integral expression for $Z(J)$ is

$$(2.1.1) \quad Z(J) = \int \mathcal{D}Q \, \exp \frac{i}{\hbar}\left[S(Q) + \int d^4x \, J_\alpha(x) Q_\alpha(x) \right]$$

where the symbol "$\mathcal{D}Q$" denotes the measure on the functional space. To explicitly show the quadratic form of $S(Q)$ one can write this as

$$(2.1.2) \quad Z(J) = \int \mathcal{D}Q \, \exp \frac{i}{\hbar} \int d^4x \left(\tfrac{1}{2} Q_\alpha(x) M_{\alpha\beta} Q_\beta(x) + J_\alpha(x) Q_\alpha(x) \right)$$

One obtains Z(J) by performing the functional integral over all field configurations Q. By analogy with simple gaussian integration, the path integral can be evaluated to give

$$Z(J) = Z(0)\exp\left[\frac{-i}{2\hbar}\int d^4x\left(J_\alpha(x) M^{-1}_{\alpha\beta} J_\beta(x)\right)\right] \tag{2.1.3}$$

where

$$Z(0) = \left(\mathrm{Det}\, M\right)^{-1/2} \tag{2.1.4}$$

and the determinant is to be taken in the functional sense. The identity

$$\det M = \exp\left[\mathrm{Tr}(\ln M)\right] \tag{2.1.5}$$

will be useful in calculations. The above evaluation of the fundamental path integral forms the basis for calculating the generating functional for theories with more general actions. A full description of functional methods and the path integral formalism can be found in many field theory textbooks {12,16}.

2.2 Definitions

Consider the simple quantum field theory of interacting scalar fields $Q_\alpha(x)$ described by an action S(Q). For notational simplicity some indices and variables will be suppressed, except where needed for illustration. Thus for example J•Q will be used to abbreviate $\int d^4x J_\alpha(x) Q_\alpha(x)$. Solutions to the theory may be obtained from a knowledge of the Green's functions

(2.2.1) $$G^n = \langle 0 | T(Q_1 Q_2 \cdots\cdots Q_n) | 0 \rangle$$

which are the vacuum expectation values of the time ordered product of the fields {12}. Using the path integral representation, the ground state vacuum amplitude for the theory in the prescence of a source J(x) is given by.

(2.2.2) $$Z(J) = \int \mathcal{D}Q \, \exp\left[\frac{i}{\hbar}\left(S(Q) + J \cdot Q \right)\right]$$

The functional Z(J) may be used to generate the Green's functions of the theory.

The complicated bookkeeping involved in calculating these Green's functions can be translated into the simpler graphical form of Feynman diagrams {12}. The Green's functions can be found by summing the graphs according to rules obtained from the full calculation with Z(J) in (2.2.2). The complexity of the diagrams, as measured by the number of loops, reflects the order in perturbation theory to which one is calculating. Those graphs in which all parts are joined are referred to as connected Feynman diagrams, and they may be summed to give the connected Green's functions.

Instead of using Z(J) from (2.2.2), it is better to work with the connected generating functional

(2.2.3) $$W(J) = -i\hbar \ln Z(J)$$

because only connected Green's functions contribute to the S-matrix {1,12}. The perturbative expansion of W(J) in powers of $\hbar$ corresponds to a loopwise expansion of connected Feynman diagrams. Hence it generates only connected Green's functions.

Next the effective action is defined by the Legendre transform

$$\Gamma(\bar{Q}) = W(J) - J \cdot \bar{Q} \tag{2.2.4}$$

where

$$\bar{Q}(x) = \frac{\delta W}{\delta J} \tag{2.2.5}$$

The effective action generates connected Green's functions which are one particle irreducible in Q. This means that their Feynman graphs cannot be made disconnected by cutting a single scalar propagator. Finally the effective potential is defined at constant field C by

$$V(C) = -\Gamma(C)\left(\int d^4x\right)^{-1} \tag{2.2.6}$$

2.3 Obtaining The Effective Potential

The procedure used by Jackiw {13} to calculate the effective potential V from the action S(Q) will be followed. One shifts the field Q(x) by an x-independent constant field C to obtain S(Q+C). Then the terms which are linear in Q are omitted. Using the definitions (2.2.2) and (2.2.3) yields the connected generating functional in the form of

$$W(J) = (\text{constant}) - i\hbar \ln \int \mathcal{D}Q \exp \frac{i}{\hbar}(\text{quadratic} + J \cdot Q)$$

$$- i\hbar \ln \left[\frac{\int \mathcal{D}Q \exp \frac{i}{\hbar}(\text{quadratic} + \text{higher order} + J \cdot Q)}{\int \mathcal{D}Q \exp \frac{i}{\hbar}(\text{quadratic} + J \cdot Q)} \right] \tag{2.3.1}$$

Note that constant, quadratic and higher order refer only to the dependence on the field Q. It has been shown by Jackiw {13} that the performance of the Legendre transform (2.2.4) on (2.3.1) gives an effective potential in the form of

$$\left(-\int d^4x\right) V = (\text{constant}) - i\hbar \ln \int \mathcal{D}Q \exp \frac{i}{\hbar}(\text{quadratic})$$

$$- i\hbar \left[\frac{\int \mathcal{D}Q \exp \frac{i}{\hbar}(\text{quadratic} + \text{higher order})}{\int \mathcal{D}Q \exp \frac{i}{\hbar}(\text{quadratic})} \right]_{1PI} \tag{2.3.2}$$

where only connected one particle irreducible graphs are to be included in the last term. Since only gaussian type path integrals can be done, the last term in (2.3.2) must be evaluated by expanding part of the exponential in a power series. The details of the procedure will become clearer when it is applied to a specific example {13,14} in the next chapter.

2.4 Dimensional Regularization

When evaluating Z(J) and the Green's functions in perturbation theory, one finds divergent integrals of the form

$$I = \int \frac{d^4 k}{(2\pi)^4} F(k) \tag{2.4.1}$$

where typically F(k) behaves for large k like k^{-2} or k^{-4}, in which case the integral has a quadratic or logarithmic divergence. To solve the problem of evaluating loop integrals such as (2.4.1) which are ultraviolet divergent, one uses the techniques of dimensional regularization. A detailed discussion of these methods is provided in a paper by 't Hooft and Veltman {19}.

To solve divergent integrals like I above, one considers

$$I' = \int \frac{d^n k}{(2\pi)^n} F(k) \tag{2.4.2}$$

which is evaluated in an arbitrary n-dimensional space. In order for the dimension of I' to be the same as I, an arbitrary scale factor μ should also be included in (2.4.2). However, physical results will be independent of this factor and the convenient choice $\mu=1$ is made. The integral I' will be finite in some domain, usually for $n<4$. It can be evaluated in this domain and the result analytically continued to include the region of interest ($n=4$). Finally the $\lim n \to 4^-$ is taken, yielding an answer which will contain poles in $(n-4)$. These must be removed by renormalization and/or cancellation with other terms.

In practice one dispenses with going through the procedure in detail each time, and uses generalized formulae if possible. The formulae for some dimensionally regularized integrals which will be needed are given in Appendix B. The integrals are over n-dimensions with the Lim $n \to 4^{-}$ understood.

III. EFFECTIVE POTENTIAL FOR THE COLEMAN-WEINBERG MODEL

In this chapter the effective potential $V(\phi)$ is calculated to $O(\hbar^2)$ for the Coleman-Weinberg model {6} discussed in Chapter One. In order to show the gauge dependence, a general Lorentz type gauge term is included in the Lagrangian density. As discussed in Chapter Two, the expansion of the effective potential in powers of $\hbar$ is related to a loop-wise expansion of the associated Feynman diagrams. The one-loop calculation is sufficient to illustrate that the effective potential is gauge dependent. The two-loop calculation is then simplified by working with the particular choice of $\alpha=0$ known as Landau gauge.

3.1 Preliminary Calculations

It is convenient to describe the complex scalar field in the model in terms of two real fields $\phi_1(x)$ and $\phi_2(x)$. Initially the Minkowski metric $g_{\mu\nu}$ $(g_{00}=1, g_{ij}=-\delta_{ij})$ will be used. The Coleman-Weinberg Lagrangian density is then given by the following expression {6}.

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}(\partial_\mu\phi_1 - eA_\mu\phi_2)^2 + \tfrac{1}{2}(\partial_\mu\phi_2 + eA_\mu\phi_1)^2 \\ &\quad - \frac{\lambda}{4!}\left(\phi_1^2+\phi_2^2\right)^2 - \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{1}{2\alpha}\left(\partial_\mu A_\mu\right)^2\end{aligned} \tag{3.1.1}$$

where $-(2\alpha)^{-1}(\partial_\mu A_\mu)^2$ is a general Lorentz type gauge term, added in order to trace the gauge dependence, in which the parameter α will fix the gauge choice.

Expanding (3.1.1) gives

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial_\mu \phi_a - \varepsilon_{ab} A_\mu \partial_\mu \phi_a \phi_b + \frac{1}{2} e^2 A_\mu A_\mu \phi_a \phi_a - \frac{\lambda}{4!}(\phi_a\phi_a)^2 - \frac{1}{2}(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu) - \frac{1}{2\alpha}(\partial_\mu A_\mu \partial_\nu A_\nu) \tag{3.1.2}$$

where ε_{ab} is the antisymmetric 2x2 matrix with $\varepsilon_{12}=1$. The action is then given by $S=\int d^4x$.

$$S(\phi_a, A_\mu) = \int d^4x \Big[\frac{1}{2}\partial_\mu \phi_a \partial_\mu \phi_a - e\varepsilon_{ab} A_\mu \partial_\mu \phi_a \phi_b + \frac{1}{2} e^2 A_\mu A_\mu \phi_a \phi_a - \frac{\lambda}{4!}(\phi_a\phi_a)^2 - \frac{1}{2}(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu) - \frac{1}{2\alpha}(\partial_\mu A_\mu \partial_\nu A_\nu)\Big] \tag{3.1.3}$$

Following the procedure {13} discussed in Chapter Two, we shift the scalar field $\phi_a(x)$ by an x-independent constant field C_a (ie. $\phi_a(x) \rightarrow \phi_a(x)+C_a$), and then omit the linear terms from the resulting expression. Note the abbreviation $C^2=C_aC_a$.

$$\begin{aligned} S(\phi_a+C_a, A_\mu) = \int d^4x \Big[& \frac{1}{2}\partial_\mu \phi_a \partial_\mu \phi_a - e\varepsilon_{ab} A_\mu \partial_\mu \phi_a C_b \\ & - e\varepsilon_{ab} A_\mu \partial_\mu \phi_a \phi_b + \frac{1}{2} e^2 A_\mu A_\mu \phi_a \phi_a + \frac{1}{2} e^2 A_\mu A_\mu C^2 - \frac{\lambda}{4!} C^4 \\ & + e^2 A_\mu A_\mu \phi_a C_a - \frac{2\lambda}{4!} C^2 \phi_a \phi_a - \frac{4\lambda}{4!} C_a C_b \phi_a \phi_b - \frac{4\lambda}{4!} C_a \phi_a \phi_b \phi_b \\ & - \frac{\lambda}{4!} \phi_a \phi_a \phi_b \phi_b - \frac{1}{2}(\partial_\mu A_\nu \partial_\mu A_\nu - \partial_\mu A_\nu \partial_\nu A_\mu) \\ & - \frac{1}{2\alpha}(\partial_\mu A_\mu \partial_\nu A_\nu)\Big] \end{aligned} \tag{3.1.4}$$

Integrating (3.1.4) by parts and dropping the surface terms gives

$$
\begin{aligned}
S(\phi_a + C_a, A_\mu) = \int d^4x \Big[& -\tfrac{1}{2}\phi_a \partial_\mu \partial_\mu \phi_a - \tfrac{1}{2} e\, \varepsilon_{ab} A_\mu \partial_\mu \phi_a C_b \\
& + \tfrac{1}{2} e\, \varepsilon_{ab} \partial_\mu A_\mu \phi_a C_b - e\, \varepsilon_{ab} A_\mu \partial_\mu \phi_a \phi_b + \tfrac{1}{2} e^2 A_\mu A_\mu \phi_a \phi_a \\
& + \tfrac{1}{2} e^2 C^2 A_\mu A_\mu + e^2 A_\mu A_\mu \phi_a C_a - \tfrac{\lambda}{4!} C^4 - \tfrac{\lambda}{12} C^2 \phi_a \phi_a \\
& - \tfrac{\lambda}{6} C_a C_b \phi_a \phi_b - \tfrac{\lambda}{6} C_a \phi_a \phi_b \phi_b - \tfrac{\lambda}{4!} \phi_a \phi_a \phi_b \phi_b \\
& + \tfrac{1}{2} A_\nu \partial_\mu \partial_\mu A_\nu - \tfrac{1}{2} A_\nu \partial_\mu \partial_\nu A_\mu + \tfrac{1}{2\alpha} A_\mu \partial_\mu \partial_\nu A_\nu \Big]
\end{aligned}
\tag{3.1.5}
$$

Collecting terms together and defining $\partial_\mu \partial_\mu = \Box$ yields

$$
\begin{aligned}
S(\phi_a + C_a, A_\mu) = \int d^4x \Big\{ & -\tfrac{\lambda}{4!} C^4 + \tfrac{1}{2}\phi_a \Big[\delta_{ab}\big(-\Box - \tfrac{\lambda}{6} C^2\big) - \tfrac{\lambda}{3} C_a C_b\Big] \phi_b \\
& + \tfrac{1}{2}\phi_a \big[e\, \varepsilon_{af} C_f \partial_\nu\big] A_\nu + \tfrac{1}{2} A_\mu \big[-\partial_\mu e\, \varepsilon_{bg} C_g\big] \phi_b - e\, \varepsilon_{ab} A_\mu \partial_\mu \phi_a \phi_b \\
& + \tfrac{1}{2} A_\mu \Big[g_{\mu\nu}\big(\Box + e^2 C^2\big) - \big(1 - \alpha^{-1}\big)\partial_\mu \partial_\nu\Big] A_\nu + e^2 A_\mu A_\mu \phi_a C_a \\
& + \tfrac{1}{2} e^2 A_\mu A_\mu \phi_a \phi_a - \tfrac{\lambda}{6} C_a \phi_a \phi_b \phi_b - \tfrac{\lambda}{4!} \phi_a \phi_a \phi_b \phi_b \Big\}
\end{aligned}
\tag{3.1.6}
$$

At this point we introduce a more compact notation for those terms in (3.1.6) which are quadratic in the fields.

Consider the row vector $Q^T(x)$

$$Q^T(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) & A_0(x) & A_1(x) & A_2(x) & A_3(x) \end{pmatrix} \tag{3.1.7}$$

and the x-representation of a 6x6 matrix M given by

$$M = \begin{pmatrix} \delta_{ab}\left(-\Box - \frac{\lambda}{6}C^2\right) - \frac{\lambda}{3}C_a C_b & e\varepsilon_{af}C_f\partial_\nu \\ & \\ -e\partial_\mu \varepsilon_{bg} C_g & g_{\mu\nu}\left(\Box + e^2C^2\right) - \left(1-\alpha^{-1}\right)\partial_\mu\partial_\nu \end{pmatrix} \tag{3.1.8}$$

With this notation (3.1.6) can be written in the form

$$\begin{aligned} S(\phi_a + C_a, A_\mu) = \int d^4x \Big[& -\frac{\lambda}{4!}C^4 + \frac{1}{2}Q^T(x) M Q(x) \\ & - e\varepsilon_{ab} A_\mu \partial_\mu \phi_a \phi_b + e^2 A_\mu A_\mu C_a \phi_a + \frac{1}{2}e^2 A_\mu A_\mu \phi_a \phi_a \\ & - \frac{\lambda}{6} C_a \phi_a \phi_b \phi_b - \frac{\lambda}{4!}\phi_a \phi_a \phi_b \phi_b \Big] \end{aligned} \tag{3.1.9}$$

Substituting (3.1.9) into (2.2.2), one obtains

$$Z = \left[\exp\frac{i}{\hbar}\int d^4x \left(-\frac{\lambda}{4!}C^4\right)\right]\left[\int \mathcal{D}\phi\,\mathcal{D}A \exp\frac{i}{2\hbar}\int d^4x\, Q^T M Q\right] Z_I \tag{3.1.10}$$

where

$$Z_I = \left\langle \exp \frac{i}{\hbar}\int d^4x \left[-e\varepsilon_{ab}A_\mu \partial_\mu \phi_a \phi_b + e^2 C_a \phi_a A_\mu A_\mu \right.\right.$$

(3.1.11)

$$\left.\left. + \frac{1}{2}e^2 A_\mu A_\mu \phi_a \phi_a - \frac{\lambda}{6} C_a \phi_a \phi_b \phi_b - \frac{\lambda}{4!}\phi_a \phi_a \phi_b \phi_b \right] \right\rangle$$

and we have defined the symbol <B> where

$$(3.1.12) \quad \langle B \rangle = \frac{\int \mathcal{D}\phi \mathcal{D}A \left[B \exp \frac{i}{2\hbar}\int d^4x\, Q^T M Q \right]}{\int \mathcal{D}\phi \mathcal{D}A \left[\exp \frac{i}{2\hbar}\int d^4x\, Q^T M Q \right]}$$

As discussed in Chapter Two, there is a correspondence between a loop-wise expansion of Feynman diagrams and an expansion of Z in powers of $\hbar$. In order to make this correspondence correctly {13}, we must rescale the fields in Z with $\phi \to \hbar^{1/2}\phi$ and $A \to \hbar^{1/2}A$. Then (3.1.11) and (3.1.12) may be written as follows.

$$Z_I = \left\langle \exp i\hbar^{1/2} \int d^4x \left[-e\varepsilon_{ab}A_\mu \partial_\mu \phi_a \phi_b + e^2 C_a \phi_a A_\mu A_\mu \right.\right.$$

(3.1.13)

$$\left.\left. + \frac{1}{2}\hbar^{1/2} e^2 A_\mu A_\mu \phi_a \phi_a - \frac{\lambda}{6} C_a \phi_a \phi_b \phi_b - \frac{\lambda}{4!}\hbar^{1/2} \phi_a \phi_a \phi_b \phi_b \right] \right\rangle$$

and

$$(3.1.14) \quad \langle B \rangle = \frac{\int \mathcal{D}\phi \mathcal{D}A \left[B \exp \frac{i}{2}\int d^4x\, Q^T M Q \right]}{\int \mathcal{D}\phi \mathcal{D}A \left[\exp \frac{i}{2}\int d^4x\, Q^T M Q \right]}$$

Now we expand the exponential of (3.1.13) in a power series. Since <odd number of fields>=0, terms in the expansion with half-integer powers of $\hbar$ will not contribute to Z_I.

$$
\begin{aligned}
Z_I = 1 + i\hbar\Big\langle \int d^4x \Big[\tfrac{1}{2} e^2 A_\mu(x) A_\mu(x)\phi_a(x)\phi_a(x) \\
-\tfrac{\lambda}{4!}\phi_a(x)\phi_a(x)\phi_b(x)\phi_b(x)\Big]\Big\rangle - \tfrac{\hbar}{2}\Big\langle \int d^4x\, d^4y \Big[-e\varepsilon_{ab} A_\mu(x)\partial^x_\mu \phi_a(x)\phi_b(x) \\
+e^2 C_a \phi_a(x) A_\mu(x) A_\mu(x) - \tfrac{\lambda}{6} C_a \phi_a(x)\phi_b(x)\phi_b(x)\Big] \\
\times\Big[-e\varepsilon_{df} A_\nu(y)\partial^y_\nu \phi_d(y)\phi_f(y) + e^2 C_d \phi_d(y) A_\nu(y) A_\nu(y) \\
-\tfrac{\lambda}{6} C_d \phi_d(y)\phi_f(y)\phi_f(y)\Big]\Big\rangle + O(\hbar^2)
\end{aligned}
\tag{3.1.15}
$$

Performing the multiplication within the double integral gives

$$
\begin{aligned}
Z_I = 1 + i\hbar\int d^4x \Big[\tfrac{1}{2} e^2 \langle A_\mu(x) A_\mu(x)\phi_a(x)\phi_a(x)\rangle - \tfrac{\lambda}{4!}\langle\phi_a(x)\phi_a(x)\phi_b(x)\phi_b(x)\rangle\Big] \\
-\tfrac{\hbar}{2}\int d^4x\, d^4y \Big[e^2 \varepsilon_{ab}\varepsilon_{df}\langle A_\mu(x)\partial^x_\mu\phi_a(x)\phi_b(x) A_\nu(y)\partial^y_\nu\phi_d(y)\phi_f(y)\rangle \\
-2e^3 \varepsilon_{ab} C_d \langle A_\mu(x)\partial^x_\mu\phi_a(x)\phi_b(x)\phi_d(y) A_\nu(y) A_\nu(y)\rangle \\
+\tfrac{\lambda}{3} e\,\varepsilon_{ab} C_d \langle A_\mu(x)\partial^x_\mu \phi_a(x)\phi_b(x)\phi_d(y)\phi_f(y)\phi_f(y)\rangle \\
+e^4 C_a C_d \langle \phi_a(x) A_\mu(x) A_\mu(x)\phi_d(y) A_\nu(y) A_\nu(y)\rangle \\
-\tfrac{\lambda}{3} e^2 C_a C_d \langle \phi_a(x) A_\mu(x) A_\mu(x)\phi_d(y)\phi_f(y)\phi_f(y)\rangle \\
+\tfrac{\lambda^2}{36} C_a C_d \langle \phi_a(x)\phi_b(x)\phi_b(x)\phi_d(y)\phi_f(y)\phi_f(y)\rangle\Big] + O(\hbar^2)
\end{aligned}
\tag{3.1.16}
$$

where the linear property $\langle A+B\rangle = \langle A\rangle + \langle B\rangle$ has been used.

We can now substitute (3.1.10) into (2.2.4) to obtain the effective action. Recall that only terms corresponding to connected diagrams are to be included from Z_I.

$$\Gamma(C) = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + O(\hbar^3) \tag{3.1.17}$$

$$\Gamma_0 = -\frac{\lambda}{4!}C^4\left(\int d^4x\right) \tag{3.1.18}$$

$$\Gamma_1 = -i\ln\left[\int \mathcal{D}\phi\mathcal{D}A \exp\frac{i}{2\hbar}\int d^4x\, Q^T M Q\right] - i \tag{3.1.19}$$

$$\Gamma_2 = -\frac{i}{\hbar}\left(Z_I - 1\right) \tag{3.1.20}$$

The effective potential may be obtained from the effective action using (2.2.6), with a similar correspondence for V_0, V_1, V_2.

$$V(C) = V_0 + \hbar V_1 + \hbar^2 V_2 + O(\hbar^3) \tag{3.1.21}$$

$$V_0 = \frac{\lambda}{4!}C^4 \tag{3.1.22}$$

$$V_1 = i\left(\int d^4x\right)^{-1}\left\{\ln\left[\int \mathcal{D}\phi\mathcal{D}A \exp\frac{i}{2\hbar}\int d^4x\, Q^T M Q\right] + 1\right\} \tag{3.1.23}$$

$$V_2 = \frac{i}{\hbar}\left(Z_I - 1\right)\left(\int d^4x\right)^{-1} \tag{3.1.24}$$

3.2 One-Loop Calculation Of V(C)

In the last section the effective potential was found to be given by the following expression.

$$V(C) = \frac{\lambda}{4!}C^4 + i\hbar\left(\int d^4x\right)^{-1} \ln\left[\int \mathcal{D}\Phi\,\mathcal{D}A \exp\frac{i}{2\hbar}\int d^4x\, Q^T M Q\right] \tag{3.2.1}$$

$$+ (\text{constant}) + \hbar^2 V_2 + O(\hbar^3)$$

where constant refers to terms independent of C. The functional integral in (3.2.1) can be expressed in terms of a functional determinant using (2.1.3).

$$V(C) = \frac{\lambda}{4!}C^4 + i\hbar\left(\int d^4x\right)^{-1}\left[-\tfrac{1}{2}\ln(\mathrm{Det}\,M)\right] + (\text{const.}) + \hbar^2 V_2 + O(\hbar^3) \tag{3.2.2}$$

The functional determinant can be expressed in terms of an ordinary determinant {13} using (2.1.5).

$$\ln(\mathrm{Det}\,M) = \left(\int d^4x\right)\int \frac{d^4k}{(2\pi)^4}\ln(\det M) \tag{3.2.3}$$

where M is expressed in the momentum representation

$$M = \begin{pmatrix} \delta_{ab}\left(k^2 - \frac{\lambda}{6}C^2\right) - \frac{\lambda}{3}C_a C_b & ie\,\varepsilon_{af}C_f k_\nu \\ \\ -ie\,k_\mu \varepsilon_{bg}C_g & -g_{\mu\nu}\left(k^2 - e^2C^2\right) + (1-\alpha^{-1})k_\mu k_\nu \end{pmatrix} \tag{3.2.4}$$

The determinant of M is evaluated in Appendix A and is given by (A.37) with r,s defined by (A.35),(A.36).

Substituting these results into (3.2.2), the effective potential expression becomes

$$V(C) = \frac{\lambda}{4!}C^4 - \frac{i\hbar}{2}\int \frac{d^4k}{(2\pi)^4} \ln\Big[(k^2-e^2C^2)^3\left(k^2-\frac{\lambda}{2}C^2\right)\left(k^2-\frac{\lambda}{12}rC^2\right) \tag{3.2.5}$$

$$\times\left(k^2-\frac{\lambda}{12}sC^2\right)\Big] + (\text{const.}) + \hbar^2 V_2 + O(\hbar^3)$$

The integral in (3.2.5) is more conveniently performed in Euclidean space. This is possible since the final result for the effective potential is unaffected by a switch from Minkowski to Euclidean space. The change is affected by now using the Euclidean metric ($g_{\mu\nu}=\delta_{\mu\nu}$), and making the substitution $k_0 \rightarrow ik_0$ in the zero component of the momentum vectors. The equivalent Euclidean space expression for (3.2.5) is then

$$V(C) = \frac{\lambda}{4!}C^4 + \frac{\hbar}{2}\int \frac{d^4k}{(2\pi)^4}\Big[3\ln(k^2+e^2C^2) + \ln\left(k^2+\frac{\lambda}{2}C^2\right) \tag{3.2.6}$$

$$+\ln\left(k^2+\frac{\lambda}{12}rC^2\right) + \ln\left(k^2+\frac{\lambda}{12}sC^2\right)\Big] + (\text{const.}) + \hbar^2 V_2 + O(\hbar^3)$$

where a minus sign from within each ln term has been factored out and included with the terms independent of C. The integrals in (3.2.6) will be solved using dimensional regularization techniques {19}. Consider the n-dimensional integral

$$I = \int \frac{d^n k}{(2\pi)^n} \ln(k^2+A) \tag{3.2.7}$$

Taking the partial derivative of (3.2.7) with respect to A gives

$$\frac{\partial I}{\partial A} = \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2+A)} \tag{3.2.8}$$

Evaluating this with (B.1) from Appendix B gives

$$\frac{\partial I}{\partial A} = \frac{\Gamma(1-n/2)}{(4\pi)^{n/2}} A^{n/2-1} \tag{3.2.9}$$

Integrating the result yields

$$I = \frac{\Gamma(1-n/2)}{(4\pi)^{n/2}} \frac{2}{n} A^{n/2} + (\text{constant}) \tag{3.2.10}$$

where the integration constant refers to terms independent of A. Substituting $n=4-2\varepsilon$

$$I = \frac{\Gamma(\varepsilon-1)}{(4\pi)^{2-\varepsilon}} \frac{1}{(2-\varepsilon)} A^{2-\varepsilon} + (\text{const.}) \tag{3.2.11}$$

Expanding (3.2.11) in a Taylor series in ε and using (B.5) to expand $\Gamma(\varepsilon-1)$ results in

$$I = \frac{-A^2}{32\pi^2}\left[\frac{1}{\varepsilon} + \psi(2) + \frac{1}{2} - \ln(A/4\pi)\right] + O(\varepsilon) + (\text{const.}) \tag{3.2.12}$$

Applying (3.2.12) to (3.2.6)

$$V(C) = \frac{\lambda}{4!}C^4 - \frac{\hbar C^4}{64\pi^2}\Big\{3e^4\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} - \ln(e^2C^2/4\pi)\right]$$

$$+ \frac{\lambda^2}{4}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} - \ln(\lambda C^2/8\pi)\right] + \frac{\lambda^2 r^2}{144}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} - \ln(\lambda r C^2/48\pi)\right] \tag{3.2.13}$$

$$+ \frac{\lambda^2 s^2}{144}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} - \ln(\lambda s C^2/48\pi)\right]\Big\} + (\text{const.}) + \hbar^2 V_2 + O(\hbar^3)$$

Since the effective potential may only be defined up to an arbitrary constant, the terms independent of C may be dropped. Noting that $r^2+s^2=4-48\alpha e^2\lambda^{-1}$, we combine terms in (3.2.13) to obtain

$$V(C) = \frac{C^4}{4!}\Big\{\lambda - \frac{\hbar}{96\pi^2}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[108e^4 - 12\alpha e^2\lambda\right.$$

$$\left.+ 10\lambda^2\right] + \frac{\hbar}{64\pi^2}\Big[72e^4\ln e^2 + 6\lambda^2\ln(\lambda/2) + \frac{\lambda^2 r^2}{6}\ln(\lambda r/12) \tag{3.2.14}$$

$$+ \frac{\lambda^2 s^2}{6}\ln(\lambda s/12) + \ln C^2\left(72e^4 + \tfrac{20}{3}\lambda^2 - 8\alpha e^2\lambda\right)\Big]\Big\}$$

$$+ \hbar^2 V_2 + O(\hbar^3)$$

3.3 One-Loop Renormalization Of V(C)

We will take the limit as $\varepsilon \to 0^+$ in (3.2.14), and hence the ε^{-1} term diverges. At this order it is sufficient to renormalize the mass and the coupling constant λ, as this will remove the $O(\hbar)$ divergent piece from (3.2.14). The usual mass and coupling constant renormalization conditions {6} for the theory are given by

$$\left.\frac{dV}{dC^2}\right|_{C=0} = \frac{m_R^2}{2} \tag{3.3.1}$$

$$\left.\frac{d^2V}{dC^4}\right|_{C=0} = \frac{\lambda_R}{12} \tag{3.3.2}$$

It is not possible in this case to define the renormalized coupling constant λ_R with (3.3.2), due to the logarithmic singularty in V(C) at C=0. Instead one must choose some arbitrary point $C=C_0$ away from zero at which to define λ_R.

$$\left.\frac{d^2V}{dC^4}\right|_{C=C_0} = \frac{\lambda_R}{12} \tag{3.3.3}$$

Substituting (3.2.14) into (3.3.1) and (3.3.3) gives

$$\begin{aligned} m_R^2 = \frac{C^2}{6}\Bigg\{ & \lambda - \frac{\hbar}{96\pi^2}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[108e^4 - 12\alpha e^2\lambda + 10\lambda^2\right] \\ & + \frac{\hbar}{64\pi^2}\Big[72e^4 \ln e^2 + 6\lambda^2 \ln(\lambda/2) + \frac{\lambda^2 r^2}{6}\ln\left(\frac{\lambda r}{12}\right) + \frac{\lambda^2 s^2}{6}\ln\left(\frac{\lambda s}{12}\right) \\ & + 36e^4 + \tfrac{10}{3}\lambda^2 - 4\alpha e^2\lambda + \ln C^2\left(72e^4 + \tfrac{20}{3}\lambda^2 - 8\alpha e^2\lambda\right)\Big]\Bigg\}\Bigg|_{C=0} = 0 \end{aligned} \tag{3.3.4}$$

and

$$\lambda_R = \lambda - \frac{\hbar}{96\pi^2}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[108e^4 - 12\alpha e^2\lambda + 10\lambda^2\right]$$

$$(3.3.5)\quad + \frac{\hbar}{64\pi^2}\left[72e^4 \ln e^2 + 6\lambda^2 \ln\left(\frac{\lambda}{2}\right) + \frac{\lambda^2 r^2}{6}\ln\left(\frac{\lambda r}{12}\right) + \frac{\lambda^2 s^2}{6}\ln\left(\frac{\lambda s}{12}\right)\right.$$

$$\left. + \ln C_0^2\left(72e^4 + \tfrac{20}{3}\lambda^2 - 8\alpha e^2\lambda\right) + 108e^4 + 10\lambda^2 - 12\alpha e^2\lambda\right]$$

Since both the bare and renormalized masses are zero, there is no need to use a mass counterterm. However the coupling constant λ differs from λ_R. Expressing λ in terms of λ_R from (3.3.5)

$$\lambda = \lambda_R + \frac{\hbar}{96\pi^2}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[108e^4 - 12\alpha e^2\lambda_R + 10\lambda_R^2\right]$$

$$(3.3.6)\quad - \frac{\hbar}{64\pi^2}\left[72e^4 \ln e^2 + 6\lambda_R^2 \ln\left(\frac{\lambda_R}{2}\right) + \frac{\lambda_R^2 r_R^2}{6}\ln\left(\frac{\lambda_R r_R}{12}\right) + \frac{\lambda_R^2 s_R^2}{6}\ln\left(\frac{\lambda_R s_R}{12}\right)\right.$$

$$\left. + \ln C_0^2\left(72e^4 + \tfrac{20}{3}\lambda_R^2 - 8\alpha e^2\lambda_R\right) + 108e^4 + 10\lambda_R^2 - 12\alpha e^2\lambda_R\right] + O(\hbar^2)$$

where r_R and s_R are merely the expressions r and s using λ_R instead of λ. Substituting (3.3.6) into (3.2.14)

$$V(C) = \frac{C^4}{4!}\left\{\lambda_R + \frac{\hbar}{64\pi^2}\left[\left(72e^4 + \tfrac{20}{3}\lambda_R^2 - 8\alpha e^2\lambda_R\right)\ln\left(C^2/C_0^2\right)\right.\right.$$

$$(3.3.7)\quad \left.\left. - 108e^4 - 10\lambda_R^2 + 12\alpha e^2\lambda_R\right]\right\} + O(\hbar^2)$$

Since λ_R was defined at an arbitrary point, one may redefine it as follows

$$\lambda_R \longrightarrow \lambda_R + \frac{\hbar}{64\pi^2}\Big[108e^4 + 10\lambda_R^2 - 12\alpha e^2\lambda_R + \left(72e^4 + \tfrac{20}{3}\lambda_R^2 - 8\alpha e^2\lambda_R\right)\ln C_0^2\Big] \tag{3.3.8}$$

Substituting (3.3.8) into (3.3.7)

$$V(C) = \frac{C^4}{4!}\left[\lambda_R + \frac{\hbar}{8\pi^2}\left(\tfrac{5}{6}\lambda_R^2 + 9e^4 - \alpha e^2\lambda_R\right)\ln C^2\right] + O(\hbar^2) \tag{3.3.9}$$

This is renormalized effective potential to $O(\hbar)$.

While the above renormalization procedure was simple enough at this level, difficulties arise at higher orders. Expressing λ in terms of λ_R from (3.3.5) to the next order would be an onerus task. In addition, the charge and the fields may need to be renormalized. Things are greatly simplified by using an alternate renormalization procedure called minimal subtraction. With this method the coupling constant in (3.2.14) is defined as a power series in $\hbar$. The divergent pieces in V(C) can then be cancelled by an appropriate choice of the coefficients in the power series. Substituting $\lambda = \lambda_0 + \hbar\lambda_1 + \hbar^2\lambda_2 + O(\hbar^3)$ into (3.2.14)

$$\begin{aligned} V(C) = \frac{C^4}{4!}\Big\{&\lambda_0 + \hbar\lambda_1 - \frac{\hbar}{96\pi^2}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[108e^4 - 12\alpha e^2\lambda_0 + 10\lambda_0^2\right] + \frac{\hbar}{64\pi^2}\Big[72e^4\ln e^2 + 6\lambda_0^2\ln\left(\frac{\lambda_0}{2}\right) + \frac{\lambda_0^2 r_0^2}{6}\ln\left(\frac{\lambda_0 r_0}{12}\right) \\ &+ \frac{\lambda_0^2 S_0^2}{6}\ln\left(\frac{\lambda_0 S_0}{12}\right) + \ln C^2\left(72e^4 + \tfrac{20}{3}\lambda_0^2 - 8\alpha e^2\lambda_0\right)\Big]\Big\} + O(\hbar^2) \end{aligned} \tag{3.3.10}$$

where r_0 and s_0 are the expressions for r and s using λ_0 instead of λ. Making the following choice for λ_1

$$\begin{aligned}\lambda_1 = {} & (96\pi^2)^{-1}\left[\varepsilon^{-1}+\psi(2)+\tfrac{1}{2}+\ln 4\pi\right]\left[108e^4+10\lambda_0^2-12\alpha e^2\lambda_0\right] \\ & -(64\pi^2)^{-1}\left[72e^4\ln e^2+6\lambda_0^2\ln\left(\frac{\lambda_0}{2}\right)+\frac{\lambda_0^2 r_0^2}{6}\ln\left(\frac{\lambda_0 r_0}{12}\right)+\frac{\lambda_0^2 s_0^2}{6}\ln\left(\frac{\lambda_0 s_0}{12}\right)\right]\end{aligned} \tag{3.3.11}$$

and substituting (3.3.11) into (3.3.10) gives

$$V(C)=\frac{C^4}{4!}\left[\lambda_0+\frac{\hbar}{8\pi^2}\left(\frac{5}{6}\lambda_0^2+9e^4-\alpha e^2\lambda_0\right)\ln C^2\right]+O(\hbar^2) \tag{3.3.12}$$

The interpretation of λ_0 in terms of the usual renormalized coupling constant may be made by comparison with (3.3.3). Since (3.3.12) is the same as (3.3.9), the two renormalization procedures are equivalent. Henceforth only the simpler minimal subtraction method will be used. When it becomes necessary, the charge and the fields will also be renormalized in this way.

Through simple algebra one may obtain the stationary points of V(C) from (3.3.12).

$$C=0\ ,\quad C=\pm\exp-\left[\frac{16\lambda_0\pi^2+\hbar\left(\frac{5}{6}\lambda_0^2+9e^4-\alpha e^2\lambda_0\right)}{4\hbar\left(\frac{5}{6}\lambda_0^2+9e^4-\alpha e^2\lambda_0\right)}\right] \tag{3.3.13}$$

where C=0 is a maximum and the other two points are minima. Since the minimum is no longer at C=0, we see that the $O(\hbar)$ correction to V(C) has indeed broken the symmetry dynamically.

3.4 Calculation Of V(C) In Landau Gauge To Two Loop

The expression for the $O(\hbar^2)$ correction to the effective action was found in Section 3.1 and is obtainable from (3.1.20) and (3.1.16).

$$\Gamma_2 = \int d^4x \left[\frac{e^2}{2} \langle A_\mu(x) A_\mu(x) \Phi_a(x) \Phi_a(x) \rangle - \frac{\lambda}{4!} \langle \Phi_a(x) \Phi_a(x) \Phi_b(x) \Phi_b(x) \rangle \right]$$

$$+ i \int d^4x\, d^4y \left[e^2 \varepsilon_{ad} \varepsilon_{bf} \langle A_\mu(x) \partial_\mu^x \Phi_a(x) \Phi_d(x) A_\nu(y) \partial_\nu^y \Phi_b(y) \Phi_f(y) \rangle \right.$$

$$- 2e^3 \varepsilon_{ad} C_b \langle A_\mu(x) \partial_\mu^x \Phi_a(x) \Phi_d(x) A_\nu(y) A_\nu(y) \Phi_b(y) \rangle$$

$$+ \frac{1}{3} \lambda e \varepsilon_{ad} C_b \langle A_\mu(x) \partial_\mu^x \Phi_a(x) \Phi_d(x) \Phi_f(y) \Phi_f(y) \Phi_b(y) \rangle \tag{3.4.1}$$

$$+ e^4 C_a C_b \langle A_\mu(x) A_\mu(x) \Phi_a(x) A_\nu(y) A_\nu(y) \Phi_b(y) \rangle$$

$$- \frac{1}{3} \lambda e^2 C_a C_b \langle A_\mu(x) A_\mu(x) \Phi_a(x) \Phi_f(y) \Phi_f(y) \Phi_b(y) \rangle$$

$$\left. + \frac{1}{36} \lambda^2 C_a C_b \langle \Phi_d(x) \Phi_d(x) \Phi_a(x) \Phi_f(y) \Phi_f(y) \Phi_b(y) \rangle \right]$$

We define the following notation

$$G_{ab}(x,y) = -i \langle \Phi_a(x) \Phi_b(y) \rangle \tag{3.4.2}$$

$$G_{\mu\nu}(x,y) = -i \langle A_\mu(x) A_\nu(y) \rangle \tag{3.4.3}$$

$$G_{\mu b}(x,y) = -i \langle A_\mu(x) \Phi_b(y) \rangle \tag{3.4.4}$$

$$G_{a\nu}(x,y) = -i \langle \Phi_a(x) A_\nu(y) \rangle \tag{3.4.5}$$

Next we expand (3.4.1) using Wick's theorem {12}. As discussed in Chapter Two, only the terms corresponding to connected diagrams which are also one particle irreducible in Φ are to be kept. Also note that $\varepsilon_{ad}\langle\Phi_a\Phi_d\rangle=0$ since ε_{ad} is antisymmetric while $\langle\Phi_a\Phi_d\rangle$ is symmetric under $a\leftrightarrow d$. Thus we have

$$\Gamma_2 = -\int d^4x\Big[\tfrac{1}{2}e^2 G_{\mu\mu}(x,x)\,G_{aa}(x,x) - \tfrac{\lambda}{4!}G_{aa}(x,x)\,G_{bb}(x,x)$$

$$-\tfrac{2\lambda}{4!}G_{ab}(x,x)\,G_{ba}(x,x)\Big] + \tfrac{1}{2}\int d^4x\,d^4y\Big\{2e^4 C_aC_b\,G_{\mu\nu}(x,y)\,G_{\nu\mu}(x,y)\,G_{ab}(x,y)$$

$$(3.4.6)\qquad + e^2\varepsilon_{ad}\varepsilon_{bf}\,G_{\mu\nu}(x,y)\Big[\big(\partial^x_\mu\partial^y_\nu G_{ab}(x,y)\big)G_{df}(x,y) + \big(\partial^x_\mu G_{af}(x,y)\big)\big(\partial^y_\nu G_{db}(x,y)\big)\Big]$$

$$+\tfrac{\lambda^2}{36}C_aC_b\Big[2G_{ab}(x,y)\,G_{df}(x,y)\,G_{fd}(x,y) + 4G_{af}(x,y)\,G_{fd}(x,y)\,G_{db}(x,y)\Big]\Big\}$$

$$+\Big(\text{terms with factors of the form } G_{\mu b}(x,y) \text{ or } G_{a\nu}(x,y)\Big)$$

Fourier transforming to momentum space

$$\Gamma_2 = -\int d^4x\int\frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\Big[\tfrac{1}{2}e^2 G_{\mu\mu}(p)\,G_{aa}(k) - \tfrac{\lambda}{4!}G_{aa}(p)\,G_{bb}(k)$$

$$-\tfrac{2\lambda}{4!}G_{ab}(p)\,G_{ba}(k) - \tfrac{1}{2}e^2\varepsilon_{ad}\varepsilon_{bf}\,G_{\mu\nu}(p+k)\,k_\mu k_\nu\,G_{ab}(k)\,G_{df}(p)$$

$$(3.4.7)\qquad -\tfrac{1}{2}e^2\varepsilon_{ad}\varepsilon_{bf}\,G_{\mu\nu}(p+k)\,k_\mu p_\nu\,G_{af}(k)\,G_{db}(p)$$

$$-e^4C_aC_b\,G_{ab}(p+k)\,G_{\mu\nu}(p)\,G_{\nu\mu}(k) - \tfrac{\lambda^2}{36}C_aC_b\,G_{ab}(p)\,G_{df}(k)\,G_{fd}(p+k)$$

$$-\tfrac{2\lambda^2}{36}C_aC_b\,G_{af}(p+k)\,G_{fd}(p)\,G_{db}(k)\Big]$$

$$+\Big[\text{terms with factors of the form } G_{\mu b}(p) \text{ or } G_{a\nu}(k)\Big]$$

Recall from (2.1.2) and (2.1.3) the following expressions.

$$Z(J) = \int \mathcal{D}Q \exp i \int d^4x \left(\tfrac{1}{2} Q^T M Q + J^T Q\right) \tag{3.4.8}$$

$$Z(J) = Z(0) \exp -\tfrac{i}{2} \int J^T M^{-1} J \, d^4x \tag{3.4.9}$$

From these one obtains respectively

$$\frac{1}{Z(0)} \cdot \frac{\delta^2 Z(J)}{\delta J_\alpha(x)\,\delta J_\beta(y)}\bigg|_{J=0} = \frac{-\int \mathcal{D}Q\, Q_\alpha(x) Q_\beta(y) \exp \frac{i}{2}\int d^4x\, Q^T M Q}{\int \mathcal{D}Q \exp \frac{i}{2} \int d^4x\, Q^T M Q} \tag{3.4.10}$$

$$\frac{1}{Z(0)} \frac{\delta^2 Z(J)}{\delta J_\alpha(x)\,\delta J_\beta(y)}\bigg|_{J=0} = -i\,\delta^4(x-y)\, M^{-1}_{\alpha\beta} \tag{3.4.11}$$

Equating (3.4.10) and (3.4.11) yields $\langle Q_\alpha(x) Q_\beta(x)\rangle = i\delta^4(x-y) M^{-1}_{\alpha\beta}$. In keeping with the previous notation this becomes $G_{\alpha\beta}(x,y) = \delta^4(x-y) M^{-1}_{\alpha\beta}$. Fourier transform to momentum space to obtain $G_{\alpha\beta}(k) = M^{-1}_{\alpha\beta}$. In this case the matrix M is the one given by (3.1.8). The momentum representation of M^{-1} is calculated in Appendix A. In $\alpha=0$ Landau gauge the result is given by

$$G_{ab}(k) = \left[\delta_{ab}\left(k^2 - \tfrac{1}{2}C^2\right) + \tfrac{1}{3} C_a C_b\right]\left(k^2 - \tfrac{1}{2}C^2\right)^{-1}\left(k^2 - \tfrac{1}{6}C^2\right)^{-1} \tag{3.4.12}$$

$$G_{\mu\nu}(k) = \left(-g_{\mu\nu} + k_\mu k_\nu k^{-2}\right)\left(k^2 - e^2 C^2\right)^{-1} \tag{3.4.13}$$

$$G_{a\nu}(k) = G_{\mu b}(k) = 0 \tag{3.4.14}$$

The $\mathcal{O}(\hbar^2)$ contribution to the effective potential V_2 may now be determined.

$$
\begin{aligned}
V_2 = \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \Big[& \tfrac{1}{2} e^2 G_{\mu\mu}(p)\, G_{aa}(k) - \tfrac{\lambda}{4!} G_{aa}(p)\, G_{bb}(k) - \tfrac{2\lambda}{4!} G_{ab}(p) G_{ba}(k) \\
& - \tfrac{1}{2} e^2 \varepsilon_{ad} \varepsilon_{bf}\, G_{\mu\nu}(p+k)\, k_\mu \left(k_\nu G_{ab}(k) G_{df}(p) + p_\nu G_{af}(k)\, G_{db}(p) \right) \\
& - e^4 C_a C_b\, G_{ab}(p+k)\, G_{\mu\nu}(p)\, G_{\nu\mu}(k) - \tfrac{\lambda^2}{36} C_a C_b\, G_{ab}(p)\, G_{df}(k)\, G_{fd}(p+k) \\
& - \tfrac{2\lambda^2}{36} C_a C_b\, G_{af}(p+k)\, G_{fd}(p)\, G_{db}(k) \Big]
\end{aligned}
\tag{3.4.15}
$$

The terms in (3.4.15) correspond to the five two-loop Feynman diagrams which follow. The integrals can be evaluated with dimensional regularization techniques.

3.5 Scalar-Photon "Figure Eight" Contribution To V(C)

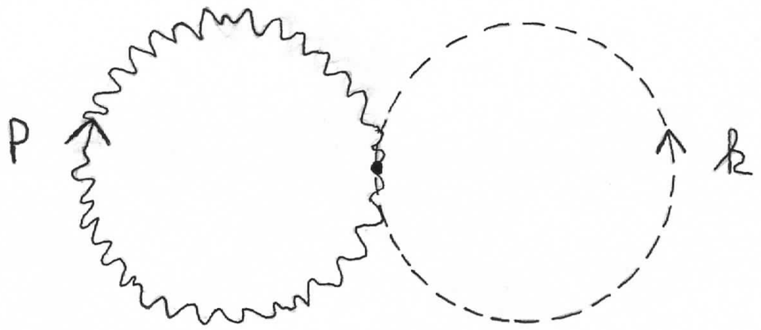

Figure 1 - The Scalar-Photon "Figure Eight" Diagram

Consider the term in (3.4.15) given by

$$\text{(3.5.1)} \quad I_1 = \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{e^2}{2} G_{\mu\mu}(p)\, G_{aa}(k)$$

which corresponds to the Feynman diagram in Figure One. Substituting (3.4.12) and (3.4.13) into (3.5.1)

$$\text{(3.5.2)} \quad I_1 = \frac{e^2}{2} \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{(1-g_{\mu\mu})}{(p^2-e^2C^2)} \left[\frac{1}{(k^2-\frac{\lambda}{2}C^2)} + \frac{1}{(k^2-\frac{\lambda}{6}C^2)} \right]$$

As discussed previously, we may evaluate the integrals using the equivalent Euclidean space expression

$$\text{(3.5.3)} \quad I_1 = \frac{(n-1)e^2}{2} \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{1}{(p^2+e^2C^2)} \left[\frac{1}{(k^2+\frac{\lambda}{2}C^2)} + \frac{1}{(k^2+\frac{\lambda}{6}C^2)} \right]$$

Evaluating the integrals with (B.1), and setting $n=4-2\varepsilon$ gives

$$\text{(3.5.4)} \quad I_1 = \frac{1}{2}(3-2\varepsilon)e^2 \frac{\Gamma^2(\varepsilon-1)}{(4\pi)^{4-2\varepsilon}} (e^2C^2)^{1-\varepsilon} \left[\left(\frac{\lambda}{2}C^2\right)^{1-\varepsilon} + \left(\frac{\lambda}{6}C^2\right)^{1-\varepsilon} \right]$$

Expanding this in a Taylor series in ε and using (B.5) to expand $\Gamma(\varepsilon-1)$ results in

$$\text{(3.5.5)} \quad I_1 = \frac{\lambda e^4 C^4}{(16\pi^2)^2} \left\{ \frac{12}{\varepsilon^2} + \frac{1}{\varepsilon}\Big[-8 + 24\psi(2) - 12\ln e^2 + 24\ln 4\pi - 9\ln(\lambda/2) - 3\ln(\lambda/6) - 24\ln C^2 \Big] \right\} + F_1(\lambda, e^2, C) + O(\varepsilon)$$

where

$$F_1(\lambda, e^2, C) = \frac{\lambda}{12} e^4 C^4 (16\pi^2)^{-2} \Big\{ \frac{9}{2} \ln^2\left(\frac{\lambda}{2}C^2\right) + \frac{3}{2} \ln^2\left(\frac{\lambda}{6}C^2\right)$$

$$+24 \ln^2(4\pi) + 6 \ln^2\left(\frac{\lambda}{2}C^2\right) + 6 \ln^2(e^2C^2) - 16 \ln(4\pi) + 8 \ln(e^2C^2)$$

$$-24 \ln(4\pi) \ln(e^2C^2) + 2 \ln^2\left(\frac{\lambda}{6}C^2\right) - 18 \ln(4\pi) \ln\left(\frac{\lambda}{2}C^2\right) \tag{3.5.6}$$

$$-6 \ln(4\pi) \ln\left(\frac{\lambda}{6}C^2\right) + 9 \ln(e^2C^2) \ln\left(\frac{\lambda}{2}C^2\right)$$

$$+3 \ln(e^2C^2) \ln\left(\frac{\lambda}{6}C^2\right) + 12\left(\frac{\pi^2}{3} + 2\psi^2(2) - \psi'(2)\right)$$

$$+2\psi(2)\left[-8 - 12 \ln(e^2C^2) + 24 \ln(4\pi) - 9 \ln\left(\frac{\lambda}{2}C^2\right) - 3 \ln\left(\frac{\lambda}{6}C^2\right)\right] \Big\}$$

3.6 Scalar-Scalar "Figure Eight" Contribution To V(C)

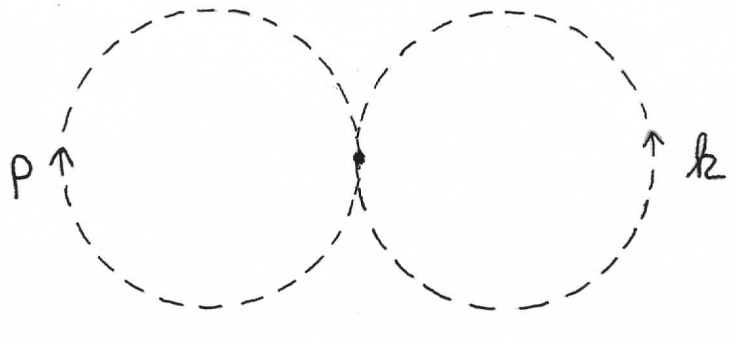

Figure 2 - The Scalar-Scalar "Figure Eight" Diagram

Consider the term in (3.4.15) given by

$$(3.6.1)\quad I_2 = -\frac{\lambda}{4!}\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\left[G_{aa}(p)\,G_{bb}(k) + 2G_{ab}(p)\,G_{ba}(k)\right]$$

which corresponds to the Feynman diagram in Figure Two. Substituting (3.4.12) into (3.6.1)

$$I_2 = -\frac{\lambda}{4!}\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\left\{\left[\left(p^2-\frac{\lambda}{2}C^2\right)^{-1}+\left(p^2-\frac{\lambda}{6}C^2\right)^{-1}\right]\left[\left(k^2-\frac{\lambda}{2}C^2\right)^{-1}\right.\right.$$

(3.6.2)

$$\left.\left.+\left(k^2-\frac{\lambda}{6}C^2\right)^{-1}\right]+2\left[\left(p^2-\frac{\lambda}{6}C^2\right)^{-1}\left(k^2-\frac{\lambda}{6}C^2\right)^{-1}+\left(p^2-\frac{\lambda}{2}C^2\right)^{-1}\left(k^2-\frac{\lambda}{2}C^2\right)^{-1}\right]\right\}$$

Changing to the equivalent Euclidean space expression for I_2 and collecting terms yields

$$I_2 = \frac{2\lambda}{4!}\left[\int\frac{d^n p}{(2\pi)^n}\left(p^2+\frac{\lambda}{2}C^2\right)^{-1}\right]\left[\int\frac{d^n k}{(2\pi)^n}\left(k^2+\frac{\lambda}{6}C^2\right)^{-1}\right]$$

(3.6.3)

$$+\frac{3\lambda}{4!}\left[\int\frac{d^n p}{(2\pi)^n}\left(p^2+\frac{\lambda}{2}C^2\right)^{-1}\right]^2+\frac{3\lambda}{4!}\left[\int\frac{d^n p}{(2\pi)^n}\left(p^2+\frac{\lambda}{6}C^2\right)^{-1}\right]^2$$

Evaluating the integrals with (B.1), and setting $n=4-2\varepsilon$ gives

$$(3.6.4)\quad I_2 = \frac{\lambda}{4!}\frac{\Gamma^2(\varepsilon-1)}{(4\pi)^{4-2\varepsilon}}\left[2\left(\frac{\lambda^2C^4}{12}\right)^{1-\varepsilon}+3\left(\frac{\lambda C^2}{2}\right)^{2-\varepsilon}+3\left(\frac{\lambda C^2}{6}\right)^{2-\varepsilon}\right]$$

Expanding in a Taylor series in ε and using (B.5) to expand $\Gamma(\varepsilon-1)$ results in

$$I_2 = \frac{\lambda^3 C^4}{4!\,(16\pi^2)^2}\left\{\frac{1}{\varepsilon^2} + \frac{1}{12\varepsilon}\left[24\psi(2) - 20\ln\left(\frac{\lambda}{2}\right) - 4\ln\left(\frac{\lambda}{6}\right)\right.\right.$$

(3.6.5)

$$\left.\left. + 24\ln(4\pi) - 24\ln(C^2)\right]\right\} + F_2(\lambda, e^2, C) + O(\varepsilon)$$

where

$$F_2(\lambda, e^2, C) = \frac{\lambda^3 C^4}{12(4!)}(16\pi^2)^{-2}\left\{24\psi^2(2) + 4\pi^2 - 12\psi'(2)\right.$$

$$+ \ln^2(\lambda^2 C^4/12) + 18\ln^2(\lambda C^2/2) + 2\ln^2(\lambda C^2/6) + 24\ln^2(4\pi)$$

(3.6.6)

$$- 40\ln(4\pi)\ln(\lambda C^2/2) - 8\ln(4\pi)\ln(\lambda C^2/6)$$

$$\left. + 2\psi(2)\left[24\ln(4\pi) - 20\ln(\lambda C^2/2) - 4\ln(\lambda C^2/6)\right]\right\}$$

3.7 Scalar "Hamburger" Contribution To V(C)

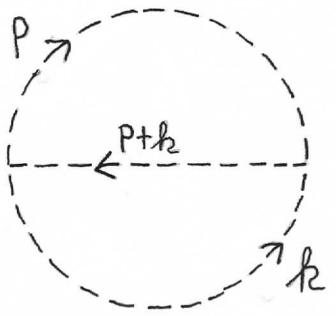

Figure 3 - The Scalar "Hamburger" Diagram

Consider the term in (3.4.15) given by

$$I_3 = -\frac{\lambda^2}{36}\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\, C_a C_b \Big[G_{ab}(p)\, G_{df}(k)\, G_{fd}(p+k) + 2\, G_{af}(p+k)\, G_{fd}(p)\, G_{db}(k)\Big] \tag{3.7.1}$$

which corresponds to the Feynman diagram in Figure Three. Substituting (3.4.12) into (3.7.1)

$$I_3 = -\frac{\lambda^2 C^2}{36}\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\Big\{ \left(p^2-\tfrac{\lambda}{2}C^2\right)^{-1}\Big[\left(k^2-\tfrac{\lambda}{6}C^2\right)^{-1}\left((p+k)^2-\tfrac{\lambda}{6}C^2\right)^{-1} + \left(k^2-\tfrac{\lambda}{2}C^2\right)^{-1}\left((p+k)^2-\tfrac{\lambda}{2}C^2\right)^{-1}\Big] + 2\left((p+k)^2-\tfrac{\lambda}{2}C^2\right)^{-1}\left(k^2-\tfrac{\lambda}{2}C^2\right)^{-1}\left(p^2-\tfrac{\lambda}{2}C^2\right)^{-1}\Big\} \tag{3.7.2}$$

Changing to the equivalent Euclidean space expression

$$I_3 = -\frac{\lambda^2 C^2}{36}\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\Big[\left((p+k)^2+\tfrac{\lambda}{6}C^2\right)^{-1}\left(k^2+\tfrac{\lambda}{6}C^2\right)^{-1}\left(p^2+\tfrac{\lambda}{2}C^2\right)^{-1} + 3\left((p+k)^2+\tfrac{\lambda}{2}C^2\right)^{-1}\left(k^2+\tfrac{\lambda}{2}C^2\right)^{-1}\left(p^2+\tfrac{\lambda}{2}C^2\right)^{-1}\Big] \tag{3.7.3}$$

Thus

$$I_3 = \frac{-\lambda^2 C^2}{36}\Big[I\left(\tfrac{\lambda}{6}C^2, \tfrac{\lambda}{6}C^2, \tfrac{\lambda}{2}C^2\right) + 3\, I\left(\tfrac{\lambda}{2}C^2, \tfrac{\lambda}{2}C^2, \tfrac{\lambda}{2}C^2\right)\Big] \tag{3.7.4}$$

where I(A,B,C) is evaluated in Appendix C and is given by

$$I(A,B,C) = \int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\left((p+k)^2+A\right)^{-1}\left(k^2+B\right)^{-1}\left(p^2+C\right)^{-1} \tag{3.7.5}$$

Substituting (C.24) into (3.7.4) gives

$$I_3 = \frac{\lambda^3 C^4}{36}(16\pi^2)^{-2}\left\{\frac{8}{3\varepsilon^2} + \frac{1}{3\varepsilon}\Big[16\psi(2) + 8 - \ln(\lambda/24\pi) - 15\ln(\lambda/8\pi) - 16\ln(C^2)\Big]\right\} + F_3(\lambda, e^2, C) + O(\varepsilon) \tag{3.7.6}$$

where

$$F_3(\lambda, e^2, C) = -\frac{\lambda^2 C^2}{36}\Big[F_I\big(\tfrac{\lambda}{6}C^2, \tfrac{\lambda}{6}C^2, \tfrac{\lambda}{2}C^2\big) + 3F_I\big(\tfrac{\lambda}{2}C^2, \tfrac{\lambda}{2}C^2, \tfrac{\lambda}{2}C^2\big)\Big] \tag{3.7.7}$$

and $F_I(A,B,C)$ is given by (C.25).

3.8 Photon "Hamburger" Contribution To V(C)

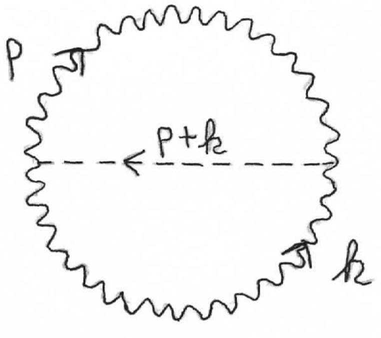

Figure 4 - The Photon "Hamburger" Diagram

Consider the term in (3.4.15) given by

$$I_4 = -e^4 \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \left[C_a C_b G_{ab}(p+k) G_{\mu\nu}(p) G_{\nu\mu}(k) \right] \tag{3.8.1}$$

which corresponds to the Feynman diagram in Figure Four. Substituting (3.4.12) and (3.4.13) into (3.8.1) gives

$$\begin{aligned} I_4 = &- e^4 C^2 \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \left[(g_{\mu\mu} - 2) + p_\mu k_\mu p_\nu k_\nu p^{-2} k^{-2} \right] \\ &\times \left((p+k)^2 - \tfrac{1}{2} C^2 \right)^{-1} \left(k^2 - e^2 C^2 \right)^{-1} \left(p^2 - e^2 C^2 \right)^{-1} \end{aligned} \tag{3.8.2}$$

Now we change to the equivalent Euclidean space expression for I_4 given by

$$\begin{aligned} I_4 = &- e^4 C^2 \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \left[(n-2) + p_\mu k_\mu p_\nu k_\nu p^{-2} k^{-2} \right] \\ &\times \left((p+k)^2 + \tfrac{1}{2} C^2 \right)^{-1} \left(k^2 + e^2 C^2 \right)^{-1} \left(p^2 + e^2 C^2 \right)^{-1} \end{aligned} \tag{3.8.3}$$

and substitute $n=4-2\varepsilon$ to obtain

$$I_4 = -e^4 C^2 \left[(2-2\varepsilon) I\left(\tfrac{1}{2}C^2, e^2C^2, e^2C^2\right) + J\left(\tfrac{1}{2}C^2, e^2C^2, e^2C^2\right) \right] \tag{3.8.4}$$

where

$$J(A,B,C) = \int \frac{d^n p}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{p_\mu p_\nu k_\mu k_\nu p^{-2} k^{-2}}{((p+k)^2 + A)(k^2 + B)(p^2 + C)} \tag{3.8.5}$$

and $J(A,B,C)$ is evaluated in Appendix D.

Substituting (C.24) and (D.31) into (3.8.4) gives

$$I_4 = e^4 c^4 (16\pi^2)^{-2} \Big\{ \tfrac{3}{4}\varepsilon^{-2}(\lambda + 3e^2) + \varepsilon^{-1}\Big[\psi(2)\left(\tfrac{3}{2}\lambda + \tfrac{9}{2}e^2\right) - \tfrac{\lambda}{8} \tag{3.8.6}$$

$$+ \tfrac{7}{8}e^2 - \tfrac{3}{2}\lambda \ln\left(\tfrac{\lambda}{8\pi}\right) - \tfrac{9}{2}e^2 \ln\left(\tfrac{e^2}{4\pi}\right) - \ln c^2 \left(\tfrac{3}{2}\lambda + \tfrac{9}{2}e^2\right)\Big]\Big\} + F_4(\lambda, e^2, c) + O(\varepsilon)$$

where

$$F_4(\lambda, e^2, c) = -e^4 c^2 \left[2 F_I\left(\tfrac{\lambda}{2}c^2, e^2c^2, e^2c^2\right) + F_J\left(\tfrac{\lambda}{2}c^2, e^2c^2, e^2c^2\right)\right] \tag{3.8.7}$$

and $F_I(A,B,C)$, $F_J(A,B,C)$ are given by (C.25), (D.32) respectively.

3.9 "Cracked Egg" Contribution To V(C)

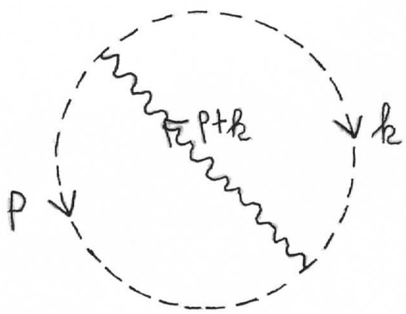

Figure 5 - The "Cracked Egg" Diagram

Consider the term in (3.4.15) given by

$$I_5 = -\tfrac{1}{2}e^2\int \frac{d^np}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}\,\varepsilon_{ad}\,\varepsilon_{bf}\,k_\mu\Big[k_\nu\,G_{\mu\nu}(p+k)\,G_{ab}(k)\,G_{df}(p) \tag{3.9.1}$$

$$+\,p_\nu\,G_{\mu\nu}(p+k)\,G_{af}(k)\,G_{dp}(p)\Big]$$

which corresponds to the Feynman diagram in Figure Five. Substituting (3.4.12) and (3.4.13) into (3.9.1) gives

$$I_5 = -\tfrac{1}{2}e^2\int \frac{d^np}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}\left[(k_\mu-p_\mu)(k_\nu-p_\nu)\left\{-g_{\mu\nu}+\frac{(p_\mu+k_\mu)(p_\nu+k_\nu)}{(p+k)^2}\right\}\right] \tag{3.9.2}$$

$$\times\;\big((p+k)^2-e^2c^2\big)^{-1}\big(k^2-\tfrac{\lambda}{6}c^2\big)^{-1}\big(p^2-\tfrac{\lambda}{2}c^2\big)^{-1}$$

Now we change to the equivalent Euclidean space expression

$$I_5 = \tfrac{1}{2}e^2\int \frac{d^np}{(2\pi)^n}\frac{d^nk}{(2\pi)^n}\left[(k_\mu-p_\mu)(k_\nu-p_\nu)\left\{-g_{\mu\nu}+\frac{(p_\mu+k_\mu)(p_\nu+k_\nu)}{(p+k)^2}\right\}\right] \tag{3.9.3}$$

$$\times\;\big((p+k)^2+e^2c^2\big)^{-1}\big(k^2+\tfrac{\lambda}{6}c^2\big)^{-1}\big(p^2+\tfrac{\lambda}{2}c^2\big)^{-1}$$

to obtain

$$I_5 = \tfrac{1}{2}e^2\,K\big(e^2c^2,\ \tfrac{\lambda}{6}c^2,\ \tfrac{\lambda}{2}c^2\big) \tag{3.9.4}$$

where

$$K(A,B,C)=\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\left[(k_\mu-p_\mu)(k_\nu-p_\nu)\left\{-g_{\mu\nu}+\frac{(p_\mu+k_\mu)(p_\nu+k_\nu)}{(p+k)^2}\right\}\right]$$

(3.9.5)

$$\times\ \left((p+k)^2+A\right)^{-1}\left(k^2+B\right)^{-1}\left(p^2+C\right)^{-1}$$

and K(A,B,C) is evaluated in Appendix E. Substituting (E.24) into (3.9.4) gives

$$I_5=\tfrac{1}{2}e^2C^4(16\pi^2)^{-2}\Big\{\tfrac{1}{2}\varepsilon^{-2}\left(e^4-2\lambda e^2-\tfrac{5}{6}\lambda^2\right)$$

(3.9.6)

$$+\varepsilon^{-1}\Big[\psi(2)\left(e^4-2\lambda e^2-\tfrac{5}{6}\lambda^2\right)+\tfrac{1}{2}\left(e^4-\tfrac{2}{3}\lambda e^2-\tfrac{7}{9}\lambda^2\right)+\tfrac{\lambda^2}{12}\ln\left(\tfrac{\lambda}{6}\right)$$

$$-e^2\left(e^2-2\lambda\right)\ln\left(e^2/4\pi\right)+\tfrac{3}{4}\lambda^2\ln\left(\tfrac{\lambda}{2}\right)-\ln C^2\left(e^4-2\lambda e^2-\tfrac{5}{6}\lambda^2\right)\Big]\Big\}$$

$$+\ F_5(\lambda,e^2,C)\ +O(\varepsilon)$$

where

(3.9.7) $$F_5(\lambda,e^2,C)=\tfrac{1}{2}e^2\,F_K\left(e^2C^2,\tfrac{\lambda}{6}C^2,\tfrac{\lambda}{2}C^2\right)$$

and $F_K(A,B,C)$ is given by (E.23).

3.10 Two-Loop Renormalization Of V(C)

The effective potential in $\alpha=0$ Landau gauge may be obtained from (3.2.14).

$$
\begin{aligned}
V(C) = \frac{C^4}{4!}\Big\{ & \lambda - \hbar(96\pi^2)^{-1}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[108e^4 + 10\lambda^2\right] \\
& + \hbar(64\pi^2)^{-1}\Big[72e^4 \ln e^2 + 6\lambda^2 \ln\left(\tfrac{\lambda}{2}\right) + \tfrac{2}{3}\lambda^2 \ln\left(\tfrac{\lambda}{6}\right) \\
& + \ln C^2\left(72e^4 + \tfrac{20}{3}\lambda^2\right)\Big]\Big\} + \hbar^2 V_2 + O(\hbar^3)
\end{aligned}
\tag{3.10.1}
$$

Substituting the results of Sections 3.5 through 3.9 into (3.4.15) gives

$$
\begin{aligned}
V_2(\lambda, e^2, C) = C^4(16\pi^2)^{-2}\Big[& \varepsilon^{-1} \ln C^2\left(-\tfrac{25}{108}\lambda^3 + \tfrac{5}{12}\lambda^2 e^2 - \tfrac{5}{2}\lambda e^4\right. \\
& \left. - 5e^6\right) + f(\lambda, e^2)\Big] + \sum_{i=1}^{5} F_i(\lambda, e^2, C)
\end{aligned}
\tag{3.10.2}
$$

The finite parts $F_i(\lambda, e^2, C)$ of V_2 have already been stated in Sections 3.5 through 3.9. The term $f(\lambda, e^2)C^4(16\pi^2)^{-2}$ is an abbreviation for the numerous divergent pieces of V_2 which have only a simple C^4 dependence on C. They are not stated explicitly here because their exact form will not be needed in the renormalization procedure.

Proceeding with the minimal subtraction method, we make the following power series expansions in $\hbar$.

$$\lambda = \lambda_0 + \hbar\lambda_1 + \hbar^2\lambda_2 + \cdots \tag{3.10.3}$$

$$e^2 = e_0^2 + \hbar e_1^2 + \hbar^2 e_2^2 + \cdots \tag{3.10.4}$$

$$C^2 = C^2(1 + \hbar z_1 + \hbar^2 z_2 + \cdots) \tag{3.10.5}$$

Substituting these expressions into (3.10.1) leads after some simple algebra to

$$\begin{aligned}
V(C) = \frac{C^4}{4!}\Big[\lambda_0 &+ \hbar\Big\{\lambda_1 + 2\lambda_0 z_1 - (96\pi^2)^{-1}\big[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} \\
&+ \ln 4\pi\big]\big[10\lambda_0^2 + 108e_0^4\big] + (64\pi^2)^{-1}\big[72e_0^4 \ln e_0^2 \\
&+ 6\lambda_0^2 \ln\left(\tfrac{\lambda_0}{2}\right) + \tfrac{2}{3}\lambda_0^2 \ln\left(\tfrac{\lambda_0}{6}\right) + \ln C^2\left(72e_0^4 + \tfrac{20}{3}\lambda_0^2\right)\big]\Big\} \\
&+ \hbar^2\Big\{\lambda_2 + 2\lambda_1 z_1 + \lambda_0(z_1^2 + 2z_2) - (96\pi^2)^{-1}\big[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} \\
&+ \ln 4\pi\big]\big[20\lambda_0\lambda_1 + 216e_0^2e_1^2 + 20z_1\lambda_0^2 + 216z_1e_0^4\big] + (64\pi^2)^{-1}\big[72e_0^2e_1^2 \\
&+ 144e_0^2e_1^2 \ln e_0^2 + 12\lambda_0\lambda_1 \ln\left(\tfrac{\lambda_0}{2}\right) + \tfrac{4}{3}\lambda_0\lambda_1 \ln\left(\tfrac{\lambda_0}{6}\right) + \tfrac{20}{3}\lambda_0\lambda_1 \\
&+ z_1\left(72e_0^4 + \tfrac{20}{3}\lambda_0^2\right) + \ln C^2\left(144e_0^2e_1^2 + \tfrac{40}{3}\lambda_0\lambda_1\right) \\
&+ 144z_1e_0^4 \ln e_0^2 + 12z_1\lambda_0^2 \ln\left(\tfrac{\lambda_0}{2}\right) + \tfrac{4}{3}z_1\lambda_0^2 \ln\left(\tfrac{\lambda_0}{6}\right) \\
&+ \ln C^2\left(144z_1e_0^4 + \tfrac{40}{3}z_1\lambda_0^2\right)\big]\Big\}\Big] + \hbar^2 V_2(\lambda_0, e_0^2, C) + O(\hbar^3)
\end{aligned} \tag{3.10.6}$$

Make the following choices for λ_1 and λ_2.

$$\lambda_1 = -2z_1\lambda_0 + (96\pi^2)^{-1}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[10\lambda_0^2 + 108e_0^4\right]$$

(3.10.7)

$$-(64\pi^2)^{-1}\left[72e_0^4 \ln e_0^2 + 6\lambda_0^2 \ln(\lambda_0/2) + \tfrac{2}{3}\lambda_0^2 \ln(\lambda_0/6)\right]$$

and

$$\lambda_2 = \lambda_2' - 2\lambda_1 z_1 - \lambda_0(z_1^2 + 2z_2) + (96\pi^2)^{-1}\left[\varepsilon^{-1} + \psi(2) + \tfrac{1}{2}\right.$$

$$\left. + \ln 4\pi\right]\left[20\lambda_0\lambda_1 + 20z_1\lambda_0^2 + 216e_0^2e_1^2 + 216z_1e_0^4\right]$$

$$-(64\pi^2)^{-1}\left[72e_0^2e_1^2 + 144e_0^2e_1^2 \ln e_0^2 + 12\lambda_0\lambda_1 \ln(\lambda_0/2)\right.$$

(3.10.8)

$$+\tfrac{4}{3}\lambda_0\lambda_1 \ln(\lambda_0/6) + \tfrac{20}{3}\lambda_0\lambda_1 + z_1(72e_0^4 + \tfrac{20}{3}\lambda_0^2)$$

$$\left. + 144z_1e_0^4 \ln e_0^2 + 12z_1\lambda_0^2 \ln\left(\tfrac{\lambda_0}{2}\right) + \tfrac{4}{3}z_1\lambda_0^2 \ln\left(\tfrac{\lambda_0}{6}\right)\right]$$

Substituting (3.10.7) and (3.10.8) into (3.10.6) gives

$$V(c) = \frac{c^4}{4!}\left\{\lambda_0 + \hbar(8\pi^2)^{-1}\left(\tfrac{5}{6}\lambda_0^2 + 9e_0^4\right)\ln c^2\right.$$

(3.10.9) $$+\hbar^2\left[\lambda_2 + \ln c^2(16\pi^2)^{-1}\left(36e_0^2e_1^2 + 36z_1e_0^4 - \tfrac{10}{3}z_1\lambda_0^2\right.\right.$$

$$\left.\left.\left. + 24\varepsilon^{-1}(16\pi^2)^{-1}\left[\tfrac{25}{108}\lambda_0^3 + \tfrac{5}{2}\lambda_0e_0^4\right]\right)\right]\right\} + \hbar^2 V_2(\lambda_0, e_0^2, c)$$

$$+\hbar^2 F_6(\lambda_0, e_0^2, C) + O(\hbar^3)$$

where

$$F_6(\lambda_0, e_0^2, C) = \frac{C^4 \ln C^2}{4!\,(16\pi^2)^2} \Big\{ \tfrac{5}{9}\left[\psi(2) + \tfrac{1}{2} + \ln 4\pi\right]\left[10\lambda_0^3 + 108\lambda_0 e_0^4\right] \tag{3.10.10}$$
$$- \tfrac{5}{6}\left(72\lambda_0 e_0^4 \ln e_0^2 + 6\lambda_0^3 \ln(\lambda_0/2) + \tfrac{2}{3}\lambda_0^3 \ln(\lambda_0/6)\right)\Big\}$$

Substituting (3.10.2) into (3.10.9)

$$V(C) = \frac{C^4}{4!}\Big\{ \lambda_0 + \hbar\,(8\pi^2)^{-1}\left(\tfrac{5}{6}\lambda_0^2 + 9e_0^4\right)\ln C^2$$
$$+ \hbar^2\Big[\lambda_2 + (16\pi^2)^{-2}\ln C^2\Big\{24\varepsilon^{-1}\left(\tfrac{5}{12}\lambda_0^2 e_0^2 - 5e_0^6\right) \tag{3.10.11}$$
$$+ (16\pi^2)\left(36e_0^2 e_1^2 + 36 z_1 e_0^4 - \tfrac{10}{3} z_1 \lambda_0^2\right)\Big\} + \frac{24}{(16\pi^2)^2} f(\lambda_0, e_0^2)\Big]\Big\}$$
$$+ \hbar^2 \sum_{i=1}^{6} F_i(\lambda_0, e_0^2, C) + O(\hbar^3)$$

Make the following choices for λ_2, z_1, e_1^2

$$\lambda_2 = \lambda_2' - 24(16\pi^2)^{-2} f(\lambda_0, e_0^2) \tag{3.10.12}$$

$$z_1 = 3e_0^2 (16\pi^2)^{-1} \varepsilon^{-1} \tag{3.10.13}$$

$$e_1^2 = e_0^4 (48\pi^2)^{-1} \varepsilon^{-1} \tag{3.10.14}$$

Putting these choices into (3.10.11) yields

$$V(C) = \frac{C^4}{4!}\left\{\lambda_0 + \hbar(8\pi^2)^{-1}\left(\frac{5}{6}\lambda_0^2 + 9e_0^4\right)\ln C^2 + \hbar^2\lambda_2\right\}$$

(3.10.15)

$$+\hbar^2\sum_{i=1}^{6} F_i(\lambda_0, e_0^2, C) + O(\hbar^3)$$

The divergent $O(\hbar^2)$ pieces have thus cancelled. The choice of λ_2 is left open so that it may be used to cancel those terms in $F_i(\lambda_0, e_0^2, C)$ which have a simple C^4 dependence on C.

3.11 Discussion Of Gauge Invariance

The one-loop renormalized effective potential was found in Section 3.3 for the Coleman-Weinberg model in a general Lorentz gauge.

$$(3.11.1)\quad V(C) = \frac{C^4}{4!}\left[\lambda + \frac{\hbar}{8\pi^2}\left(\frac{5}{6}\lambda^2 + 9e^4 - \alpha e^2\lambda\right)\ln C^2\right] + O(\hbar^2)$$

From the presence of the gauge parameter α in (3.11.1), it can be seen that the effective potential is gauge dependent. The question has been raised {7,13,14} as to how this will affect the physical consequences of the model. Any physical quantities obtained should a priori be independent of gauge choice. There can be no direct physical interpretation of a gauge dependent effective potential, and thus the validity of any approximation to the complete V(C) must be looked at.

The leading term of the scalar-vector mass ratio for the Coleman-Weinberg model has been calculated by Kang {14} at the two-loop level in a general Lorentz gauge. To the extent that the approximations made have the same behaviour for all gauge choices, the result is gauge independent as expected.

One assumes that it is possible to work with the effective potential in a convenient gauge to obtain valid physical results. Although the intermediate steps may be gauge dependent, the a priori assumption of gauge invariance for physical quantities will ensure that the final result is correct.

IV. FINITE TEMPERATURE EFFECTIVE POTENTIAL

This chapter will study the temperature dependence at the one-loop level of the effective potential for the Coleman-Weinberg model just looked at. For simplicity the $\alpha=0$ Landau gauge is worked in. The Boltzmann constant k_B will be set equal to one. The Euclidean metric $(g_{\mu\nu}=\delta_{\mu\nu})$ is used and the temperature parameter will be the inverse temperature $\beta=T^{-1}$. The main result is to show that the broken symmetry of the theory is restored at high temperature.

4.1 Changes From The Zero Temperature Case

The method of calculating the effective potential $V(\beta^{-1})$ closely parallels that done in Chapter Three for $V(\beta^{-1}=0)$. One begins with the partition function

$$(4.1.1) \quad Z_\beta(J) = \frac{\mathrm{Tr}\left[\exp(-\beta H)\, T\exp\left(\frac{i}{\hbar}\int d^4x\, J\cdot Q\right)\right]}{\mathrm{Tr}\left[\exp(-\beta H)\right]}$$

where the trace is taken over all possible states of the system described by the Hamiltonian H. With this exception, the definitions of Section 2.2 are unaltered. The calculation then proceeds as described in Section 2.3 with only a few changes. A detailed description of the finite temperature path integral formalism may be found in the literature {3,8,11}.

For calculational purposes, one need only note the following few simple results. In the finite temperature case one is restricted to the section of Euclidean space bounded by $0<t<\beta$. The volume element thus becomes

$$\int d^4x \longrightarrow \int_0^\beta dt \int d^3x \tag{4.1.2}$$

The boson fields defined on this space must satisfy the periodic boundary conditions

$$A_\mu(\beta,\vec{x}) = A_\mu(0,\vec{x}) \tag{4.1.3}$$

$$\Phi_a(\beta,\vec{x}) = \Phi_a(0,\vec{x}) \tag{4.1.4}$$

The time integration is now over a finite range and hence the Fourier transform to momentum space is discrete rather than continuous. This results in the following changes from the zero temperature formalism.

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \beta^{-1} \sum_n \int \frac{d^3k}{(2\pi)^3} \tag{4.1.5}$$

$$k_0 \longrightarrow \omega_n = 2\pi n \beta^{-1} \tag{4.1.6}$$

With these results one may now calculate the effective potential at finite temperature. Note that for zero temperature $\beta \rightarrow \infty$ and the results will reduce to those in Chapter Three.

4.2 Finite Temperature Calculation Of V(C)

Recall from (3.2.6) that the one-loop effective potential for $\alpha=0$ Landau gauge is given by

(4.2.1)
$$V(\beta^{-1}=0) = \frac{\lambda C^4}{4!} + \frac{\hbar}{2}\int \frac{d^4k}{(2\pi)^4}\Big[3\ln(k^2+e^2C^2) + \ln(k^2+\tfrac{\lambda}{2}C^2) + \ln(k^2+\tfrac{\lambda}{6}C^2)\Big] + O(\hbar^2)$$

for the $\beta^{-1}=0$ case. Noting the changes described in Section 4.1, the result for the general case is then given by

(4.2.2)
$$V(\beta^{-1}) = \frac{\lambda C^4}{4!} + \frac{\hbar}{2\beta}\sum_n\int \frac{d^3k}{(2\pi)^3}\Big[3\ln(\vec{k}^2+e^2C^2+\omega_n^2) + \ln(\vec{k}^2+\tfrac{\lambda C^2}{2}+\omega_n^2) + \ln(\vec{k}^2+\tfrac{\lambda C^2}{6}+\omega_n^2)\Big] + O(\hbar^2)$$

Consider the integral

(4.2.3)
$$I(\beta^{-1}) = \beta^{-1}\int \frac{d^3k}{(2\pi)^3}\, D$$

where

(4.2.4)
$$D = \sum_n \ln(\vec{k}^2 + A^2 + \omega_n^2)$$

Taking the partial derivative of D with respect to $\vec{k}^2$ gives

(4.2.5)
$$\frac{\partial D}{\partial \vec{k}^2} = \sum_n (\vec{k}^2 + A^2 + \omega_n^2)^{-1}$$

The series (4.2.5) may be found in reference tables {10} and summed to yield

$$\frac{dD}{d\vec{k}^2} = \frac{\beta^2}{4}\,\frac{\mathrm{cth}\left[\frac{\beta}{2}(\vec{k}^2+A^2)^{1/2}\right]}{\frac{\beta}{2}(\vec{k}^2+A^2)^{1/2}} \tag{4.2.6}$$

Integrating gives

$$D = 2\ln\left(\sinh\left[\frac{\beta}{2}(\vec{k}^2+A^2)^{1/2}\right]\right) + D' \tag{4.2.7}$$

with a constant of integration D'. Writing sinh in terms of exponentials in (4.2.7) and substituting into (4.2.3) gives

$$\begin{aligned} I(\beta^{-1}) = \beta^{-1}\int \frac{d^3k}{(2\pi)^3}\Big[&\beta(\vec{k}^2+A^2)^{1/2} - 2\ln 2 \\ &+ D' + 2\ln\left(1-\exp\{-\beta(\vec{k}^2+A^2)^{1/2}\}\right)\Big] \end{aligned} \tag{4.2.8}$$

The term involving ln2 may be dropped as it is independent of A^2. At zero temperature $I(\beta^{-1})$ in (4.2.8) must reduce to its equivalent $I(\beta^{-1}=0)$ calculated in Section 3.2. Thus (4.2.8) becomes

$$I(\beta^{-1}) = I(\beta^{-1}=0) + 2\beta^{-1}\int \frac{d^3k}{(2\pi)^3}\ln\left\{1-\exp\left[-\beta(\vec{k}^2+A^2)^{1/2}\right]\right\} \tag{4.2.9}$$

Finally the integrations over angles may be performed and the variable change $x^2 = \beta^2 k^2$ made to yield

$$I(\beta^{-1}) = I(\beta^{-1}=0) + \pi^2 \beta^{-4} \int_0^\infty dx\ x^2 \ln\left\{1-\exp\left[-(x^2+\beta^2 A^2)^{1/2}\right]\right\} \tag{4.2.10}$$

Substituting (4.2.10) into (4.2.2) to obtain the finite temperature effective potential gives

$$\begin{aligned} V(\beta^{-1}) = V(\beta^{-1}=0) + \frac{\hbar \beta^{-4}}{2\pi^2} \int_0^\infty dx\ x^2 \Big[& 3\ln\left\{1-\exp\left[-(x^2+\beta^2 e^2 C^2)^{1/2}\right]\right\} \\ & + \ln\left\{1-\exp\left[-(x^2+\beta^2 \tfrac{\lambda}{2} C^2)^{1/2}\right]\right\} \\ & + \ln\left\{1-\exp\left[-(x^2+\beta^2 \tfrac{\lambda}{6} C^2)^{1/2}\right]\right\} \quad + O(\hbar^2) \end{aligned} \tag{4.2.11}$$

The term $V(\beta^{-1}=0,C)$ was found in Chapter Three and is given by (3.10.15). The x-integrations in (4.2.11) cannot be evaluated exactly and some approximation technique must be used.

4.3 High Temperature Expansion

Consider the integral piece of (4.2.10) in the high temperature limit as $\beta \to 0$.

$$I' = \frac{\beta^{-4}}{2\pi^2} \int_0^\infty dx\ x^2 \left\{1 - \exp\left[-(x^2+\beta^2 A^2)^{1/2}\right]\right\} \tag{4.3.1}$$

One may expand the integrand in a Taylor series about $\beta=0$, and subsequently perform the integration over x. Dolan and Jackiw have discussed this expansion in detail {8}, and their result is given by

$$I' = -\frac{\pi^2\beta^{-4}}{90} + \frac{A^2\beta^{-2}}{24} - \frac{A^3\beta^{-1}}{12\pi} - \frac{A^4\ln(\beta^2A^2)}{64\pi^2}$$

(4.3.2)

$$+\frac{\left(\frac{3}{2}+2\ln 4\pi-2\gamma\right)A^4}{64\pi^2} + O(\beta^2A^6)$$

Substituting (4.3.2) into (4.2.11) and retaining only the leading high temperature terms yields

$$V(\beta^{-1}) = V(\beta^{-1}=0) + \hbar\left[-\frac{\pi^2\beta^{-4}}{18} + \frac{\beta^{-2}C^2}{24}\left(3e^2+\tfrac{2}{3}\lambda\right)+\cdots\right] \quad (4.3.3)$$

The β^{-4} term is independent of C and may be dropped. Thus the finite temperature effective potential at the one-loop level for high temperature is

$$V(\beta^{-1}) = V(\beta^{-1}=0) + \frac{\hbar\beta^{-2}C^2}{24}\left(3e^2+\tfrac{2}{3}\lambda\right) + \cdots \quad (4.3.4)$$

4.4 Symmetry Restoration

Recall the zero temperature effective potential from equation (3.3.12) for $\alpha=0$ Landau gauge.

$$V(\beta^{-1}=0) = \frac{C^4}{4!}\left[\lambda + \frac{\hbar}{8\pi^2}\left(\tfrac{5}{6}\lambda+9e^4\right)\ln C^2\right] + O(\hbar^2) \quad (4.4.1)$$

Also recall its stationary points from (3.3.13)

$$C = 0 \tag{4.4.2}$$

$$C = \pm \exp - \left[\frac{16\lambda\pi^2 + \hbar\left(\frac{5}{6}\lambda + 9e^4\right)}{4\hbar\left(\frac{5}{6}\lambda + 9e^4\right)} \right] \tag{4.4.3}$$

As discussed in Chapter Three, the prescence of the two stationary points other than C=0 reflects the broken symmetry of the model at zero temperature.

At high temperature the effective potential from (4.3.4) has its dominant C-dependent behaviour in the form of

$$V(\beta^{-1}) = \hbar\beta^{-2}C^2\left(3e^2 + \tfrac{2}{3}\lambda\right)/24 \tag{4.4.4}$$

and C=0 is the only stationary point. Hence the symmetry is restored at high temperature. From (4.3.4) it can be seen that the first derivative of $V(\beta^{-1},C)$ is continuous, so the phase transition from the symmetric to the broken case is not first order.

V. SUMMARY AND DISCUSSION

This chapter summarizes the effective potential techniques just developed. It then considers their application to studying models other than Coleman-Weinberg, concentrating in particular on SU(n) gauge theories.

5.1 Summary Of The Calculation

The effective potential was introduced using the path integral formalism in Chapter Two. The method of its calculation was then outlined for the general case. A knowledge of the effective potential allows one to derive many physical results.

The details involved in the calculation were given by example through the study of the Coleman-Weinberg model in Chapter Three. Evaluating the one-loop renormalized effective potential demonstrated the dynamical symmetry breaking of the model, and also indicated that the result was gauge dependent. It was concluded that the a priori assumption of having gauge invariant physical quantities should ensure that one can work with the effective potential method in any convenient gauge and still obtain the correct physical results. The two-loop calculation in Landau gauge was then performed which explicitly demonstrated the renormalizibility of the effective potential to that order.

Finally in Chapter Four the temperature dependence of the effective potential was determined to one loop order. The important result was that the symmetry of the model which was dynamically broken at T=0 could be restored at high temperatures. It is this two phase nature, where a dynamically broken gauge theory can have its symmetry restored, that will be of great use when applying effective potential methods to other models.

5.2 Applications To SU(n) Gauge Theories

The techniques developed above may be used to study more complex theories than Coleman-Weinberg. Particularly interesting are SU(n) gauge theories describing quarks which interact through gluon gauge fields. Both quarks and gluons carry an extra quantum number known as colour in addition to the usual set of quantum numbers. Since an individual quark has never been observed, it is believed that only colour neutral states exist. The quarks are bound in such a way that the resulting colour quantum number of the state is zero. At sufficiently high temperature, it is thought that the quarks could undergo a phase transition to an unconfined phase of free quarks. That such a transition is possible has been demonstrated in lattice calculations of quark confinement at high temperature {15,18}. Thermal fluctuations cause the gluon fields to screen the confining forces between the quarks. It is possible to show {15} that the confining phase and the deconfining phase correspond to states of the system where a symmetry of the underlying theory is, respectively, intact or broken. This symmetry is the Z(n) symmetry, where Z(n) is the centre of the SU(n) gauge group of transformations (i.e. $\Omega \rightarrow \exp(\pm 2\pi i/n)\Omega$). At high enough temperature the Z(n) symmetry is broken and the quarks are free in an unconfined phase.

What has just been described is a gauge theory characterized by a high temperature phase with a broken symmetry (deconfining) and another low temperature phase where the symmetry is intact (confining). The similarities to the Coleman-Weinberg model suggest that the effective potential techniques reviewed above could be used to study the confining-deconfining phase transition in the continuum for SU(n) gauge theories.

As a specific case we take the SU(2) gauge theory which is described by the partition function

$$Z = \int \mathcal{D}(g\beta A_0(\vec{x}))\, dA_i^a(\vec{x},t)\, \exp\left[-\frac{1}{2}\int_0^\beta dt \int d^3x\, (E^2+B^2)\right] \tag{5.2.1}$$

where

$$E^2 = \left[\partial_0 A_i^a - \partial_i A_0^a + g(A_0 \times A_i)^a\right]^2 \tag{5.2.2}$$

and

$$B^2 = \frac{1}{2}\left[\partial_i A_j^a - \partial_j A_i^a + g(A_i \times A_j)^a\right]^2 \tag{5.2.3}$$

with $a,i,j=1,2,3$. Working from (5.2.1) in a gauge $A_0^a(x)=\delta_{a3}\Phi(\vec{x})$, and using the same techniques shown in the

Coleman-Weinberg example, Weiss {20} calculated the finite temperature one-loop effective potential obtaining

$$V(C) = -2\pi^2\beta^{-4}\left[\frac{1}{45} - \frac{1}{24}\left\{1 - \left(\frac{BC}{\pi} - 1\right)^2\right\}^2\right] + \ldots \tag{5.2.4}$$

This perturbation calculation is valid at high temperature and shows that the symmetry is broken. The two-loop calculation has not yet been done, and it is hoped that doing so might aid in the understanding of how the symmetry is restored at lower temperatures. The feasability of such a study is currently being investigated.

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APPENDIX A - INVERSE AND DETERMINANT OF THE MATRIX M

In this appendix the inverse and determinant will be calculated for a matrix M in the momentum representation. In what follows k^2 is defined using the Minkowski metric $g_{\mu\nu}$ ($g_{00}=1, g_{ij}=-\delta_{ij}$) where $\mu,\nu=0,1,2,3$. The definition $C^2=C_1^2+C_2^2$ is also used. First consider the 2x2 matrix F given by

$$F_{ab} = \delta_{ab}\left(k^2 - \frac{\lambda}{6}C^2\right) - \frac{\lambda}{3}C_a C_b \tag{A.1}$$

where a,b=1,2 and δ is the Kronecker delta function. The determinant and inverse of F are then given by

$$\det F = \left(k^2 - \frac{\lambda}{2}C^2\right)\left(k^2 - \frac{\lambda}{6}C^2\right) \tag{A.2}$$

$$F_{ab}^{-1} = \frac{\delta_{ab}\left(k^2 - \frac{\lambda}{2}C^2\right) + \frac{\lambda}{3}C_a C_b}{\left(k^2 - \frac{\lambda}{2}C^2\right)\left(k^2 - \frac{\lambda}{6}C^2\right)} \tag{A.3}$$

Consider the 4x4 matrix N given by

$$N_{\mu\nu} = -g_{\mu\nu}\left(k^2 - A\right) + B k_\mu k_\nu \tag{A.4}$$

where $\mu,\nu=0,1,2,3$. The inverse and determinant of N are then given by

$$\det N = -\left(k^2 - A\right)^3\left(k^2 - Bk^2 - A\right) \tag{A.5}$$

$$N_{\mu\nu}^{-1} = \frac{-g_{\mu\nu}\left(k^2 - Bk^2 - A\right) - B k_\mu k_\nu}{\left(k^2 - Bk^2 - A\right)\left(k^2 - A\right)} \tag{A.6}$$

Consider the nxn matrix

(A.7) $$M = \begin{pmatrix} A & B \\ E & D \end{pmatrix}$$

for arbitrary, invertible matrices A,B,E,D. Let us write the inverse of M as

(A.8) $$M^{-1} = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

The equation $MM^{-1}=I$ leads to the following four conditions.

(A.9) $$W = (A - BD^{-1}E)^{-1}$$

(A.10) $$Z = (D - EA^{-1}B)^{-1}$$

(A.11) $$X = -A^{-1}BZ$$

(A.12) $$Y = -D^{-1}EW$$

Take the following specific case for M

(A.13) $$A_{ab} = \delta_{ab}\left(k^2 - \tfrac{\lambda}{6}C^2\right) - \tfrac{\lambda}{3}C_a C_b$$

(A.14) $$B_{a\nu} = i e\, \varepsilon_{af} C_f k_\nu$$

(A.15) $$E_{\mu b} = -i e\, k_\mu \varepsilon_{bg} C_g$$

(A.16) $$D_{\mu\nu} = -g_{\mu\nu}\left(k^2 - e^2C^2\right) + \left(1 - \alpha^{-1}\right) k_\mu k_\nu$$

where a,b,f,g=1,2 and $\mu,\nu=0,1,2,3$.

Now A^{-1} and D^{-1} can be found from (A.3) and (A.6) respectively. The result is

$$A^{-1}_{ab} = \frac{\delta_{ab}\left(k^2 - \frac{\lambda}{2}C^2\right) + \frac{\lambda}{3} C_a C_b}{\left(k^2 - \frac{\lambda}{2}C^2\right)\left(k^2 - \frac{\lambda}{6}C^2\right)} \tag{A.17}$$

$$D^{-1}_{\mu\nu} = \frac{-g_{\mu\nu}\left(k^2 - \alpha e^2 C^2\right) + (1-\alpha) k_\mu k_\nu}{\left(k^2 - \alpha e^2 C^2\right)\left(k^2 - e^2 C^2\right)} \tag{A.18}$$

Combining (A.13) through (A.18)

$$(A - BD^{-1}E)_{ab} = \delta_{ab}\left(k^2 - \frac{\lambda}{6}C^2\right) - \frac{\lambda}{3} C_a C_b + \frac{\alpha e^2 k^2 \varepsilon_{af}\varepsilon_{bg} C_f C_g}{\left(k^2 - \alpha e^2 C^2\right)} \tag{A.19}$$

$$(D - EA^{-1}B)_{\mu\nu} = -g_{\mu\nu}\left(k^2 - e^2 C^2\right) + (1 - \alpha^{-1}) k_\mu k_\nu - \frac{e^2 C^2 k_\mu k_\nu}{\left(k^2 - \frac{\lambda}{6}C^2\right)} \tag{A.20}$$

The determinants may be evaluated to give

$$\det(A - BD^{-1}E) = \left(k^2 - \frac{\lambda}{2}C^2\right)\left(k^2 - \alpha e^2 C^2\right)^{-1} K \tag{A.21}$$

$$\det(D - EA^{-1}B) = -\alpha^{-1}\left(k^2 - e^2 C^2\right)^3 \left(k^2 - \frac{\lambda}{6}C^2\right)^{-1} K \tag{A.22}$$

where

$$K = \left(k^2 - \frac{\lambda}{6}C^2\right)\left(k^2 - \alpha e^2 C^2\right) + \alpha e^2 C^2 k^2 \tag{A.23}$$

Using (A.19) through (A.23) one may obtain the inverses W and Z defined by (A.9) and (A.10). The results are as follows

(A.24) $$W_{ab} = \frac{(k^2 - \alpha e^2 C^2)}{(k^2 - \frac{\lambda}{6} C^2)} \left[\delta_{ab}(k^2 - \tfrac{\lambda}{2} C^2) + \tfrac{\lambda}{3} C_a C_b + \frac{\alpha e^2 k^2 C_a C_b}{(k^2 - \alpha e^2 C^2)} \right] K^{-1}$$

(A.25) $$Z_{\mu\nu} = \frac{(-g_{\mu\nu} k^2 + k_\mu k_\nu) K - \alpha (k^2 - \frac{\lambda}{6} C^2)(k^2 - e^2 C^2) k_\mu k_\nu}{k^2 K (k^2 - e^2 C^2)}$$

Substituting the expressions for A,D,B,E,Z,W into (A.11) and (A.12) gives

(A.26) $$X_{a\nu} = i e \alpha \, \varepsilon_{af} C_f k_\nu K^{-1}$$

(A.27) $$Y_{\mu b} = -i e \alpha \, k_\mu \varepsilon_{bg} C_g K^{-1}$$

The inverse of M is now given by (A.8) with W,X,Y,Z given by (A.24) through (A.27). Note that for α=0 the results reduce to the simple form

(A.28) $$W_{ab}(\alpha=0) = \left[\delta_{ab}(k^2 - \tfrac{\lambda}{2} C^2) + \tfrac{\lambda}{3} C_a C_b \right] (k^2 - \tfrac{\lambda}{2} C^2)^{-1} (k^2 - \tfrac{\lambda}{6} C^2)^{-1}$$

(A.29) $$Z_{\mu\nu}(\alpha=0) = (-g_{\mu\nu} + k_\mu k_\nu / k^2)(k^2 - e^2 C^2)^{-1}$$

(A.30) $$X_{a\nu}(\alpha=0) = Y_{\mu b}(\alpha=0) = 0$$

Finally the determinant of M may be calculated. Recall the following matrix algebra property {17}.

(A.31) $$\det M = \det \begin{pmatrix} A & B \\ E & D \end{pmatrix} = (\det A) \det(D - E A^{-1} B)$$

Substituting into (A.31) from (A.2) and (A.22) gives

(A.32) $$\det M = -\alpha^{-1}(k^2-e^2C^2)^3\left(k^2-\tfrac{\lambda}{6}C^2\right)^{-1}K\left(k^2-\tfrac{\lambda}{2}C^2\right)\left(k^2-\tfrac{\lambda}{6}C^2\right)$$

(A.33) $$\det M = -\alpha^{-1}(k^2-e^2C^2)^3\left(k^2-\tfrac{\lambda}{2}C^2\right)K$$

Now define

(A.34) $$r = 1+(1-24\alpha e^2\lambda^{-1})^{1/2}$$

(A.35) $$s = 1-(1-24\alpha e^2\lambda^{-1})^{1/2}$$

Using the definition of K in (A.23), the determinant of M is given by

(A.36) $$\det M = -\alpha^{-1}(k^2-e^2c^2)^3\left(k^2-\tfrac{\lambda}{2}c^2\right)\left(k^2-\tfrac{\lambda}{12}rc^2\right)\left(k^2-\tfrac{\lambda}{12}sc^2\right)$$

APPENDIX B - SOME FORMULAE REQUIRED IN THE TEXT

Below are the formulae for some dimensionally regularized integrals which use the Euclidean metric ($g_{\mu\nu}=\delta_{\mu\nu}$). They are taken from Ramond {16}. The methods for obtaining these formulae are explained in a paper by 't Hooft and Veltman {19}.

(B.1)
$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{1}{(k^2+M^2+2k\cdot p)^A} = \frac{\Gamma(A-\omega)}{\Gamma(A)(4\pi)^{\omega}} \frac{1}{(M^2-p^2)^{A-\omega}}$$

(B.2)
$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu}{(k^2+M^2+2k\cdot p)^A} = \frac{(4\pi)^{-\omega}}{\Gamma(A)}\left[\frac{p_\mu p_\nu \Gamma(A-\omega)}{(M^2-p^2)^{A-\omega}} + \frac{\frac{1}{2}\delta_{\mu\nu}\Gamma(A-\omega-1)}{(M^2-p^2)^{A-\omega-1}}\right]$$

(B.3)
$$\int \frac{d^{2\omega}k}{(2\pi)^{2\omega}} \frac{k_\mu k_\nu k_\rho k_\sigma}{(k^2+M^2+2k\cdot p)^A} = \frac{(4\pi)^{-\omega}}{\Gamma(A)}\left[\frac{p_\mu p_\nu p_\rho p_\sigma \Gamma(A-\omega)}{(M^2-p^2)^{A-\omega}}\right.$$
$$+\frac{1}{2}(\delta_{\mu\nu}p_\rho p_\sigma + \delta_{\nu\sigma}p_\mu p_\rho + \delta_{\rho\sigma}p_\mu p_\nu + \delta_{\mu\rho}p_\nu p_\sigma + \delta_{\nu\rho}p_\mu p_\sigma + \delta_{\mu\sigma}p_\nu p_\rho)\frac{\Gamma(A-\omega-1)}{(M^2-p^2)^{A-\omega-1}}$$
$$\left.+\frac{1}{4}(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma})\Gamma(A-\omega-2)(M^2-p^2)^{-A+\omega+2}\right]$$

Also taken from Ramond {16} is the formula

(B.4)
$$\left(\prod_1^n D_j^{a_j}\right)^{-1} = \frac{\Gamma(\sum_1^n a_j)}{\prod_1^n \Gamma(a_j)} \int_0^1 \cdots \int_0^1 d\alpha_1 \cdots d\alpha_n \frac{\delta(1-\sum_1^n \alpha_j)\prod_1^n \alpha_j^{a_j-1}}{(\sum_1^n D_j\alpha_j)^{a_1+a_2+\cdots+a_n}}$$

where the α_j are known as Feynman parameters.

The expansion of the Γ-function {16} about its poles at non-positive integers is given by

$$\text{(B.5)} \quad \Gamma(-m+\varepsilon) = \frac{(-1)^m}{m!}\left\{\frac{1}{\varepsilon} + \psi(m+1) + \frac{\varepsilon}{2}\left[\frac{\pi^2}{3} + \psi^2(m+1) - \psi'(m+1)\right] + O(\varepsilon^2)\right\}$$

where, using the Euler constant γ, we define ψ by

$$\text{(B.6)} \quad \psi(m+1) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{m} \quad , \quad \psi(1) = -\gamma$$

APPENDIX C - EVALUATION OF I(A,B,C)

In this appendix we evaluate the integral

$$(C.1)\quad I(A,B,C) = \int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\left((p+k)^2+A\right)^{-1}\left(k^2+B\right)^{-1}\left(p^2+C\right)^{-1}$$

with p^2, k^2, and $(p+k)^2$ defined using the Euclidean metric ($g_{\mu\nu}=\delta_{\mu\nu}$). Introducing Feynman parameters from (B.4) gives

$$(C.2)\quad I = 2\int_0^1 d\alpha_1\int_0^1 d\alpha_2\int_0^1 d\alpha_3\int\frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)\,(\alpha_1+\alpha_3)^{-3}}{\left[p^2+\dfrac{2\alpha_1 p\cdot k}{(\alpha_1+\alpha_3)}+\dfrac{A\alpha_1+B\alpha_2+C\alpha_3+k^2(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_3)}\right]^3}$$

Applying (B.1) to the integration over p

$$(C.3)\quad \begin{aligned} I = {} & \frac{\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}}\int_0^1 d\alpha_1\int_0^1 d\alpha_2\int_0^1 d\alpha_3\,\frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)(\alpha_1+\alpha_3)^{3-n}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{3-n/2}} \\ & \times\int\frac{d^n k}{(2\pi)^n}\left[k^2+\frac{(A\alpha_1+B\alpha_2+C\alpha_3)(\alpha_1+\alpha_3)}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}\right]^{-3+n/2}\end{aligned}$$

Applying (B.1) to the integration over k

$$(C.4)\quad I = \frac{\Gamma(3-n)}{(4\pi)^n}\int_0^1 d\alpha_1\int_0^1 d\alpha_2\int_0^1 d\alpha_3\,\frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)(A\alpha_1+B\alpha_2+C\alpha_3)^{n-3}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{n/2}}$$

Substituting $n=4-2\varepsilon$

(C.5) $$I = \frac{\Gamma(2\varepsilon-1)}{(4\pi)^{4-2\varepsilon}} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \; \delta(\alpha_1+\alpha_2+\alpha_3-1) \frac{(A\alpha_1+B\alpha_2+C\alpha_3)^{1-2\varepsilon}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{2-\varepsilon}}$$

Note that the integrand contains poles in the $\alpha_1, \alpha_2, \alpha_3$ parameter space. The integrals can be evaluated by following the technique of Elias and Mann {9}.

Consider the pole where $\alpha_1=\alpha_2=0$ and $\alpha_3=1$ and define, for small positive ε

(C.6) $$P_{12} = \int_0^\varepsilon d\alpha_1 \int_0^\varepsilon d\alpha_2 \int_{1-\varepsilon}^1 d\alpha_3 \; \delta(\alpha_1+\alpha_2+\alpha_3-1) \frac{(A\alpha_1+B\alpha_2+C\alpha_3)^{1-2\varepsilon}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{2-\varepsilon}}$$

Changing variables $\alpha_1'=\varepsilon\alpha_1, \alpha_2'=\varepsilon\alpha_2, \alpha_3'=1-\varepsilon\alpha_3$ and using the identity $\delta(ax)=a^{-1}\delta(x)$ gives

(C.7) $$P_{12} = \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \; \varepsilon^{\varepsilon} \delta(\alpha_1+\alpha_2-\alpha_3) \frac{\left[C+\varepsilon(A\alpha_1+B\alpha_2-C\alpha_3)\right]^{1-2\varepsilon}}{\left[\alpha_1\alpha_2\varepsilon+(\alpha_1+\alpha_2)-\varepsilon\alpha_3(\alpha_1+\alpha_2)\right]^{2-\varepsilon}}$$

where the α_i' have been rewritten as α_i. After applying binomial expansions to (C.7) and expanding $\varepsilon^{\varepsilon}$ in a Taylor series, one has

(C.8) $$P_{12} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \; \delta(\alpha_1+\alpha_2-\alpha_3) \, C^{1-2\varepsilon} (\alpha_1+\alpha_2)^{\varepsilon-2}$$

With the variable change $\alpha_1=\rho x, \alpha_2=\rho(1-x)$ (C.8) becomes

$$\text{(C.9)} \quad P_{12} \xrightarrow{\varepsilon\to 0} \int_0^1 dx \int_0^1 \rho\, d\rho \int_0^1 d\alpha_3\, \delta(\rho-\alpha_3)\, C^{1-2\varepsilon} \rho^{\varepsilon-2} = \varepsilon^{-1} C^{1-2\varepsilon}$$

An alternate form of the pole P_{12} will be needed. By making the substitution $\alpha_3' = 1-\alpha_3$ (C.8) becomes

$$\text{(C.10)} \quad P_{12} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \delta(\alpha_1+\alpha_2+\alpha_3-1)\, C^{1-2\varepsilon} (\alpha_1+\alpha_2)^{\varepsilon-2}$$

The leading $\varepsilon\to 0$ behaviour of the poles at $\alpha_1=\alpha_3=0, \alpha_2=1$ and $\alpha_2=\alpha_3=0, \alpha_1=1$ can be obtained from (C.5) by a similar derivation. More simply, one can relabel the integration variables in (C.6) and obtain the results below by comparison with (C.9) and (C.10). The two alternate forms for each pole are given by

$$\text{(C.11)} \quad P_{13} \xrightarrow{\varepsilon\to 0} \varepsilon^{-1} B^{1-2\varepsilon}$$

$$\text{(C.12)} \quad P_{13} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \delta(\alpha_1+\alpha_2+\alpha_3-1)\, B^{1-2\varepsilon} (\alpha_1+\alpha_3)^{\varepsilon-2}$$

$$\text{(C.13)} \quad P_{23} \xrightarrow{\varepsilon\to 0} \varepsilon^{-1} A^{1-2\varepsilon}$$

$$\text{(C.14)} \quad P_{23} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \delta(\alpha_1+\alpha_2+\alpha_3-1)\, A^{1-2\varepsilon} (\alpha_2+\alpha_3)^{\varepsilon-2}$$

One can rewrite (C.5) by adding the poles in the form of (C.9),(C.11),(C.13) and subtracting them in the form of (C.10),(C.12),(C.14).

This results in

$$I = \frac{\Gamma(2\varepsilon-1)}{(4\pi)^{4-2\varepsilon}} \left[\frac{A^{1-2\varepsilon} + B^{1-2\varepsilon} + C^{1-2\varepsilon}}{\varepsilon} + \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \, \delta(\alpha_1 + \alpha_2 + \alpha_3 - 1) \right.$$

(C.15)

$$\left. \times \left\{ \frac{(A\alpha_1 + B\alpha_2 + C\alpha_3)^{1-2\varepsilon}}{(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)^{2-\varepsilon}} - \frac{A^{1-2\varepsilon}}{(\alpha_2+\alpha_3)^{2-\varepsilon}} - \frac{B^{1-2\varepsilon}}{(\alpha_1+\alpha_3)^{2-\varepsilon}} - \frac{C^{1-2\varepsilon}}{(\alpha_1+\alpha_2)^{2-\varepsilon}} \right\} \right]$$

Changing variables $\alpha_1 = \rho x, \alpha_2 = \rho(1-x)$ and performing the α_3 integration gives

$$I = \frac{\Gamma(2\varepsilon-1)}{(4\pi)^{4-2\varepsilon}} \left[\frac{A^{1-2\varepsilon} + B^{1-2\varepsilon} + C^{1-2\varepsilon}}{\varepsilon} + \int_0^1 dx \int_0^1 \rho \, d\rho \right.$$

(C.16)

$$\left. \times \left\{ \frac{[A\rho x + B\rho(1-x) + C(1-\rho)]^{1-2\varepsilon}}{[\rho(1+a\rho)]^{2-\varepsilon}} - \frac{A^{1-2\varepsilon}}{(1-\rho x)^{2-\varepsilon}} - \frac{B^{1-2\varepsilon}}{[1-\rho(1-x)]^{2-\varepsilon}} - \frac{C^{1-2\varepsilon}}{\rho^{2-\varepsilon}} \right\} \right]$$

where we define a=x(1-x)-1. Performing a Taylor expansion in ε inside the integral results in the expression

$$I = \frac{\Gamma(2\varepsilon-1)}{(4\pi)^{4-2\varepsilon}} \left[\frac{A^{1-2\varepsilon} + B^{1-2\varepsilon} + C^{1-2\varepsilon}}{\varepsilon} + \int_0^1 dx \int_0^1 \rho \, d\rho \right.$$

(C.17)

$$\times \left\{ \frac{A\rho x + B\rho(1-x) + C(1-\rho)}{\rho^2(1+a\rho)^2} - \frac{A}{(1-\rho x)^2} - \frac{B}{[1-\rho(1-x)]^2} - \frac{C}{\rho^2} \right\}$$

$$\left. + \varepsilon \, G(A,B,C) + O(\varepsilon^2) \right]$$

where

$$G(A,B,C) = \int_0^1 dx \int_0^1 \rho\, d\rho \left[\left[\frac{A\rho x + B\rho(1-x) + C(1-\rho)}{\rho^2 (1+a\rho)^2} \right] \left\{ \ln[\rho(1+a\rho)] \right. \right.$$

$$\text{(C.18)} \quad \left. - 2\ln[A\rho x + B\rho(1-x) + C(1-\rho)] \right\} - A(1-\rho x)^{-2}[\ln(1-\rho x) - 2\ln A]$$

$$\left. - B[1-\rho(1-x)]^{-2}[\ln(1-\rho(1-x)) - 2\ln B] - C\rho^{-2}[\ln\rho - 2\ln C] \right]$$

With the aid of integral tables {10} consider

$$C\int_0^1 dx \int_0^1 d\rho \left[(1+a\rho)^{-2} - 1 \right] = -C\int_0^1 dx \left[\ln(1+a) + a(1+a)^{-1} \right]$$

$$\text{(C.19)} \qquad = C + 2C\int_0^1 dx\, x^{-1}$$

$$\int_0^1 dx \int_0^1 d\rho \left[\frac{Ax + B(1-x) - C}{(1+a\rho)^2} \right] = \int_0^1 dx \left[\frac{Ax + B(1-x) - C}{(1+a)} \right]$$

$$\text{(C.20)} \qquad = (A+B-2C)\int_0^1 dx\, x^{-1}$$

$$\text{(C.21)} \quad \int_0^1 dx \int_0^1 d\rho \left[-A\rho(1-\rho x)^{-2} - B\rho(1-\rho(1-x))^{-2} \right] = -(A+B)\int_0^1 d\rho\, \rho(1-\rho)^{-1}$$

$$= A + B - (A+B)\int_0^1 d\rho\, \rho^{-1}$$

Substituting (C.19),(C.20),(C.21) into (C.17)

$$\text{(C.22)} \quad I = \frac{\Gamma(2\varepsilon - 1)}{(4\pi)^{4-2\varepsilon}} \left[\frac{A^{1-2\varepsilon} + B^{1-2\varepsilon} + C^{1-2\varepsilon}}{\varepsilon} + A + B + C + \varepsilon G(A,B,C) + O(\varepsilon^2) \right]$$

Now perform a Taylor expansion in ε on all but $\Gamma(2\varepsilon-1)$

$$I = \frac{\Gamma(2\varepsilon-1)}{(16\pi^2)^2}\left[\frac{A+B+C}{\varepsilon} + \left\{A+B+C-2A\ln\left(\frac{A}{4\pi}\right)-2B\ln\left(\frac{B}{4\pi}\right)-2C\ln\left(\frac{C}{4\pi}\right)\right\}\right.$$

(C.23)

$$+\varepsilon\left\{G(A,B,C)+2A\ln^2 A+2B\ln^2 B+2C\ln^2 C+2(A+B+C)\ln^2 4\pi\right.$$

$$\left.\left.+2\ln 4\pi(A+B+C-2A\ln A-2B\ln B-2C\ln C)\right\} + O(\varepsilon^2)\right]$$

Using (B.5) to expand $\Gamma(2\varepsilon-1)$ finally yields

$$I(A,B,C) = \frac{-1}{(16\pi^2)^2}\left[\frac{A+B+C}{2\varepsilon^2} + \frac{1}{2\varepsilon}\left\{2(A+B+C)\psi(2) + A+B+C\right.\right.$$

(C.24)

$$\left.\left.-2A\ln\left(\frac{A}{4\pi}\right)-2B\ln\left(\frac{B}{4\pi}\right)-2C\ln\left(\frac{C}{4\pi}\right)\right\}\right] + F_I(A,B,C) + O(\varepsilon)$$

where

$$F_I(A,B,C) = -(16\pi^2)^{-2}\left[\frac{1}{2}G(A,B,C) + A\ln^2 A + B\ln^2 B + C\ln^2 C\right.$$

$$+(A+B+C)\ln^2 4\pi + \ln 4\pi(A+B+C-2A\ln A-2B\ln B-2C\ln C)$$

(C.25)

$$+\psi(2)\left\{A+B+C-2A\ln(A/4\pi)-2B\ln(B/4\pi)-2C\ln(C/4\pi)\right\}$$

$$\left.+(A+B+C)\left\{\frac{\pi^2}{3}+\psi^2(2)-\psi'(2)\right\}\right]$$

APPENDIX D - EVALUATION OF J(A,B,C)

In this appendix we evaluate the integral

$$(\mathrm{D}.1)\qquad J(A,B,C)=\int \frac{d^np}{(2\pi)^n}\,\frac{d^nk}{(2\pi)^n}\,\frac{(p\cdot k)^2/p^2k^2}{((p+k)^2+A)(k^2+B)(p^2+C)}$$

where the Euclidean metric ($g_{\mu\nu}=\delta_{\mu\nu}$) has been used. Consider the identity

$$(\mathrm{D}.2)\qquad 1=B^{-1}C^{-1}\left[p^2k^2+(k^2+B)(p^2+C)-p^2(k^2+B)-k^2(p^2+C)\right]$$

Inserting (D.2) into (D.1) one obtains

$$(\mathrm{D}.3)\qquad \begin{aligned} J(A,B,C)= B^{-1}C^{-1}\int \frac{d^np}{(2\pi)^n}\,\frac{d^nk}{(2\pi)^n}&\left[1+(k^2+B)(p^2+C)p^{-2}k^{-2}-(k^2+B)k^{-2}\right.\\ &\left.-(p^2+C)p^{-2}\right]\left[((p+k)^2+A)(k^2+B)(p^2+C)\right]^{-1}(p\cdot k)^2 \end{aligned}$$

Thus

$$(\mathrm{D}.4)\qquad J(A,B,C)=B^{-1}C^{-1}\left[J_1(A,B,C)+J_1(A,0,0)-J_1(A,0,C)-J_1(A,B,0)\right]$$

where

$$(\mathrm{D}.5)\qquad J_1(A,B,C)=\int \frac{d^np}{(2\pi)^n}\,\frac{d^nk}{(2\pi)^n}\,(p\cdot k)^2\left[((p+k)^2+A)(k^2+B)(p^2+C)\right]^{-1}$$

Introducing Feynman parameters with (B.4) gives

$$J_1(A,B,C) = 2\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\,\delta(\alpha_1+\alpha_2+\alpha_3-1)\,(\alpha_1+\alpha_3)^{-3}$$

(D.6)

$$\times\, p_\mu p_\nu k_\mu k_\nu \left[p^2 + \frac{2\alpha_1\, p\cdot k}{(\alpha_1+\alpha_3)} + \frac{A\alpha_1 + B\alpha_2 + C\alpha_3 + k^2(\alpha_1+\alpha_2)}{(\alpha_1+\alpha_3)}\right]^{-3}$$

Applying (B.2) to the integration over p

$$J_1(A,B,C) = \frac{1}{(4\pi)^{n/2}}\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)\,(\alpha_1+\alpha_3)^{3-n}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{3-n/2}}\int\frac{d^n k}{(2\pi)^n}$$

(D.7) $$\times\left\{\frac{\alpha_1^2 k^4\,\Gamma(3-\frac{n}{2})}{(\alpha_1+\alpha_3)^2}\left[k^2 + \frac{(A\alpha_1+B\alpha_2+C\alpha_3)(\alpha_1+\alpha_3)}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}\right]^{\frac{n}{2}-3}\right.$$

$$\left. + \frac{\frac{1}{2}k^2\,\Gamma(2-\frac{n}{2})(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}{(\alpha_1+\alpha_3)^2}\left[k^2 + \frac{(A\alpha_1+B\alpha_2+C\alpha_3)(\alpha_1+\alpha_3)}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}\right]^{\frac{n}{2}-2}\right\}$$

Applying (B.2) and (B.3) to the integration over k

$$J_1(A,B,C) = \frac{n\,\Gamma(1-n)}{4(4\pi)^n}\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \delta(\alpha_1+\alpha_2+\alpha_3-1)(A\alpha_1+B\alpha_2+C\alpha_3)^{n-1}$$

(D.8)

$$\times\left\{\left[(n+2)\alpha_1^2 + (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\right]\left[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3\right]^{-2-\frac{n}{2}}\right\}$$

Substituting (D.8) into (D.4)

$$J_1(A,B,C) = \frac{n\,\Gamma(1-n)}{4BC\,(4\pi)^n}\int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{\varepsilon-4}}\Big[(6-2\varepsilon)\alpha_1^2$$

(D.9)

$$+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\Big]\left[(A\alpha_1+B\alpha_2+C\alpha_3)^{3-2\varepsilon} + (A\alpha_1)^{3-2\varepsilon} - (A\alpha_1+C\alpha_3)^{3-2\varepsilon} - (A\alpha_1+B\alpha_2)^{3-2\varepsilon}\right]$$

Putting $n=4-2\varepsilon$ gives

$$J_1(A,B,C) = \frac{(4-2\varepsilon)\Gamma(2\varepsilon-3)}{4BC\,(4\pi)^{4-2\varepsilon}} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \; \delta(\alpha_1+\alpha_2+\alpha_3-1)$$

(D.10)
$$\times\left[(6-2\varepsilon)\alpha_1^2+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)\right]\left[\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3\right]^{\varepsilon-4}$$

$$\times\left[(A\alpha_1+B\alpha_2+C\alpha_3)^{3-2\varepsilon}+(A\alpha_1)^{3-2\varepsilon}-(A\alpha_1+C\alpha_3)^{3-2\varepsilon}-(A\alpha_1+B\alpha_2)^{3-2\varepsilon}\right]$$

As in Appendix C the integrals can be solved by following the technique of Elias and Mann {9}.

Consider the pole where $\alpha_1=\alpha_2=0$, $\alpha_3=1$ and define for small positive ε

$$P_{12} = \int_0^{\varepsilon} d\alpha_1 \int_0^{\varepsilon} d\alpha_2 \int_{1-\varepsilon}^{1} d\alpha_3 \; \delta(\alpha_1+\alpha_2+\alpha_3-1)\left[\frac{(6-2\varepsilon)\alpha_1^2+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{4-\varepsilon}}\right]$$

(D.11)

$$\times\left[(A\alpha_1+B\alpha_2+C\alpha_3)^{3-2\varepsilon}+(A\alpha_1)^{3-2\varepsilon}-(A\alpha_1+C\alpha_3)^{3-2\varepsilon}-(A\alpha_1+B\alpha_2)^{3-2\varepsilon}\right]$$

Changing variables $\alpha_1'=\varepsilon\alpha_1$, $\alpha_2'=\varepsilon\alpha_2$, $\alpha_3'=1-\varepsilon\alpha_3$ and using the identity $\delta(ax)=a^{-1}\delta(x)$ gives

$$P_{12} = \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3 \; \varepsilon^{\varepsilon}\, \delta(\alpha_1+\alpha_2-\alpha_3)\left[\frac{(6-2\varepsilon)\varepsilon\alpha_1^2+\varepsilon\alpha_1\alpha_2+(\alpha_1+\alpha_2)(1-\varepsilon\alpha_3)}{\varepsilon\left(\varepsilon\alpha_1\alpha_2+(\alpha_1+\alpha_2)(1-\varepsilon\alpha_3)\right)^{4-\varepsilon}}\right]$$

(D.12)

$$\times\Big[\left(A\varepsilon\alpha_1+B\varepsilon\alpha_2+C(1-\varepsilon\alpha_3)\right)^{3-2\varepsilon}+(A\varepsilon\alpha_1)^{3-2\varepsilon}-\left(A\varepsilon\alpha_1+C(1-\varepsilon\alpha_3)\right)^{3-2\varepsilon}$$
$$-(A\varepsilon\alpha_1+B\varepsilon\alpha_2)^{3-2\varepsilon}\Big]$$

where the α_i' have been rewritten as α_i. Applying binomial expansions to (D.12) and expanding $\varepsilon^{\varepsilon}$ in a Taylor series, one obtains the following expression for the leading term in ε.

$$\text{(D.13)}\quad P_{12} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\ \delta(\alpha_1+\alpha_2-\alpha_3)\left[\frac{(3-2\varepsilon)BC^{2-2\varepsilon}\alpha_2}{(\alpha_1+\alpha_2)^{3-\varepsilon}}\right]$$

With the change of variables $\alpha_1=\rho x$, $\alpha_2=\rho(1-x)$ equation (D.13) becomes

$$\text{(D.14)}\quad P_{12} \xrightarrow{\varepsilon\to 0} (3-2\varepsilon)BC^{2-2\varepsilon}\int_0^1 dx\int_0^1 \rho\, d\rho\int_0^1 d\alpha_3\ \delta(\rho-\alpha_3)\left[\frac{\rho(1-x)}{\rho^{3-\varepsilon}}\right] = \frac{(3-2\varepsilon)BC^{2-2\varepsilon}}{2\varepsilon}$$

An alternate form of P_{12} will be needed. Setting $\int_0^1 dx = 2\int_0^1 dx(1-x)$ in (D.14), changing $\alpha_3'=1-\alpha_3$ and changing variables back to $\alpha_1=\rho x$, $\alpha_2=\rho(1-x)$ gives

$$\text{(D.15)}\quad P_{12} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\ \delta(\alpha_1+\alpha_2+\alpha_3-1)\left[\frac{\frac{1}{2}(3-2\varepsilon)BC^{2-2\varepsilon}}{(\alpha_1+\alpha_2)^{2-\varepsilon}}\right]$$

One can relabel the integration variables in (D.11) and obtain the leading $\varepsilon\to 0$ behaviour for the pole at $\alpha_1=\alpha_3=0$, $\alpha_2=1$ by comparison with (D.14) and (D.15). The two alternate forms are

$$\text{(D.16)}\quad P_{13} \xrightarrow{\varepsilon\to 0} \frac{1}{2\varepsilon}(3-2\varepsilon)CB^{2-2\varepsilon}$$

$$\text{(D.17)}\quad P_{13} \xrightarrow{\varepsilon\to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\ \delta(\alpha_1+\alpha_2+\alpha_3-1)\left[\frac{\frac{1}{2}(3-2\varepsilon)CB^{2-2\varepsilon}}{(\alpha_1+\alpha_3)^{2-\varepsilon}}\right]$$

The leading $\varepsilon \to 0$ behaviour of the pole at $\alpha_2 = \alpha_3 = 0$, $\alpha_1 = 1$ follows from (D.10) by a similar derivation, with the result

$$P_{23} \xrightarrow{\varepsilon \to 0} (6\varepsilon)^{-1}(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)\,A^{1-2\varepsilon}BC \tag{D.18}$$

$$P_{23} \xrightarrow{\varepsilon \to 0} \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \delta(\alpha_1+\alpha_2+\alpha_3-1) \left[\frac{(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)A^{1-2\varepsilon}BC}{6(\alpha_2+\alpha_3)^{2-\varepsilon}}\right] \tag{D.19}$$

One can rewrite (D.10) by adding the poles in the form of (D.14),(D.16),(D.18) and subtracting them in the form of (D.15),(D.17),(D.19). This will result in the following expression

$$\begin{aligned}
J(A,B,C) = \frac{(4-2\varepsilon)\Gamma(2\varepsilon-3)}{4BC\,(4\pi)^{4-2\varepsilon}} \Bigg[& \frac{(3-2\varepsilon)}{2\varepsilon}\left(BC^{2-2\varepsilon} + CB^{2-2\varepsilon}\right) \\
& + \frac{(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)A^{1-2\varepsilon}BC}{6\varepsilon} + \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \int_0^1 d\alpha_3\, \delta(\alpha_1+\alpha_2+\alpha_3-1) \\
& \times \Bigg\{ \left[\frac{(6-2\varepsilon)\alpha_1^2 + (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{4-\varepsilon}}\right] \Big[(A\alpha_1+B\alpha_2+C\alpha_3)^{3-2\varepsilon} + (A\alpha_1)^{3-2\varepsilon} \\
& - (A\alpha_1+C\alpha_3)^{3-2\varepsilon} - (A\alpha_1+B\alpha_2)^{3-2\varepsilon} \Big] - \frac{1}{3}(3-2\varepsilon)BC^{2-2\varepsilon}(\alpha_1+\alpha_2)^{\varepsilon-2} \\
& - \frac{1}{2}(3-2\varepsilon)CB^{2-2\varepsilon}(\alpha_1+\alpha_3)^{\varepsilon-2} - \frac{1}{6}(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)BCA^{1-2\varepsilon}(\alpha_2+\alpha_3)^{\varepsilon-2} \Bigg\} \Bigg]
\end{aligned} \tag{D.20}$$

Changing variables $\alpha_2 = px$, $\alpha_3 = p(1-x)$ and doing the integration over α_1 gives

$$
\begin{aligned}
J(A,B,C) = \frac{(4-2\varepsilon)\Gamma(2\varepsilon-3)}{4BC(4\pi)^{4-2\varepsilon}} \Bigg[& \frac{(3-2\varepsilon)}{2\varepsilon}\left(BC^{2-2\varepsilon} + CB^{2-2\varepsilon}\right) + \frac{(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)BCA^{1-2\varepsilon}}{6\varepsilon} \\
& + \int_0^1 dx \int_0^1 p\,dp \Bigg\{ \left[\frac{(6-2\varepsilon)(1-p)^2 + p(1+ap)}{(p(1+ap))^{4-\varepsilon}}\right]\Big[\left(A(1-p)+Bpx+Cp(1-x)\right)^{3-2\varepsilon} + \left(A(1-p)\right)^{3-2\varepsilon} \\
& - \left(A(1-p)+Cp(1-x)\right)^{3-2\varepsilon} - \left(A(1-p)+Bpx\right)^{3-2\varepsilon}\Big] - \tfrac{1}{2}(3-2\varepsilon)BC^{2-2\varepsilon}\left(1-p(1-x)\right)^{\varepsilon-2} \\
& - \tfrac{1}{2}(3-2\varepsilon)CB^{2-2\varepsilon}(1-px)^{\varepsilon-2} - \tfrac{1}{6}(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)BCA^{1-2\varepsilon}p^{\varepsilon-2} \Bigg\} \Bigg]
\end{aligned}
\tag{D.21}
$$

where again $a=x(1-x)-1$. Performing a Taylor expansion in ε inside the integral results in

$$
\begin{aligned}
J(A,B,C) = \frac{(4-2\varepsilon)\Gamma(2\varepsilon-3)}{4(4\pi)^{4-2\varepsilon}} \Bigg[& \frac{(3-2\varepsilon)}{2\varepsilon}\left(B^{1-2\varepsilon} + C^{1-2\varepsilon}\right) + \frac{(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)A^{1-2\varepsilon}}{6\varepsilon} \\
& + 3\int_0^1 dx \int_0^1 dp \Bigg\{ 2A\left[\frac{6x(1-x)(1-p)^3}{p(1+ap)^4} - \frac{1}{p}\right] + 2A\left[\frac{x(1-x)(1-p)}{(1+ap)^3}\right] \\
& + 6\left(Bx+C(1-x)\right)\left[\frac{x(1-x)(1-p)^2}{(1+ap)^4}\right] + x(1-x)\left(Bx+C(1-x)\right)p\,(1+ap)^{-3} \\
& - \tfrac{1}{2}pC\left(1-p(1-x)\right)^{-2} - \tfrac{1}{2}pB(1-px)^{-2} \Bigg\} + \varepsilon H(A,B,C) + O(\varepsilon^2) \Bigg]
\end{aligned}
\tag{D.22}
$$

where
$$H(A,B,C)=\frac{1}{BC}\int_0^1 dx\int_0^1 p\,dp\left\{\left[\frac{6(1-p)^2+p(1+ap)}{p^4(1+ap)^4}\right]\Big[-2\big(A(1-p)\big)^3\ln\big(A(1-p)\big)\right.$$
$$-2\big(A(1-p)+Bpx+Cp(1-x)\big)^3\ln\big(A(1-p)+Bpx+Cp(1-x)\big)+2\big(A(1-p)+Bpx\big)^3\ln\big(A(1-p)+Bpx\big)$$
$$+2\big(A(1-p)+Cp(1-x)\big)^3\ln\big(A(1-p)+Cp(1-x)\big)\Big]+3BC\,p^2x(1-x)\left[\frac{-2(1-p)^2}{p^4(1+ap)^4}\right.$$

(D.23)
$$\left.+\left\{\frac{6(1-p)^2+p(1+ap)}{p^4(1+ap)^4}\right\}\ln\big(p(1+ap)\big)\right]\big(2A(1-p)+Bpx+Cp(1-x)\big)+12ABCp^{-2}$$
$$+12ABCp^{-2}\ln A-6ABCp^{-2}\ln p+\frac{B^2C+3B^2C\ln B-\frac{3}{2}B^2C\ln(1-px)}{(1-px)^2}$$
$$\left.+\big(1-p(1-x)\big)^{-2}\Big[BC^2+3BC^2\ln C-\tfrac{3}{2}BC^2\ln\big(1-p(1-x)\big)\Big]\right\}$$

With the aid of integral tables {10} consider the following

(D.24) $$2A\int_0^1 dx\int_0^1 dp\left[\frac{6x(1-x)(1-p)^3}{p(1+ap)^4}-\frac{1}{p}\right]=2A\int_0^1 dx\left[-6(1+a)\ln(1+a)-11(1+a)\right]=-\frac{A}{3}$$

(D.25) $$2A\int_0^1 dx\int_0^1 dp\left[\frac{x(1-x)(1-p)}{(1+ap)^3}\right]=2A\int_0^1 dx\left(\frac{1}{2}\right)=A$$

(D.26) $$\int_0^1 dx\int_0^1 dp\left[\frac{6\big(Bx+C(1-x)\big)x(1-x)(1-p)^2}{(1+ap)^4}\right]=\int_0^1 dx\;2\big(Bx+C(1-x)\big)=B+C$$

(D.27) $$\int_0^1 dx\int_0^1 dp\left[\frac{x(1-x)\big(Bx+C(1-x)\big)p}{(1+ap)^3}\right]=\int_0^1 dx\left[\frac{Bx+C(1-x)}{2x(1-x)}\right]=\frac{1}{2}(B+C)\int_0^1 dx\,x^{-1}$$

$$-\frac{1}{2}C\int_0^1 dx\int_0^1 d\rho\left[\frac{\rho}{[1-\rho(1-x)]^2}\right] = -\frac{1}{2}C\int_0^1 d\rho\left[\frac{\rho}{1-\rho}\right] = \frac{1}{2}C - \frac{1}{2}C\int_0^1 d\rho\,\rho^{-1} \tag{D.28}$$

$$-\frac{1}{2}B\int_0^1 dx\int_0^1 d\rho\left[\frac{\rho}{(1-\rho x)^2}\right] = -\frac{1}{2}B\int_0^1 d\rho\left[\frac{\rho}{1-\rho}\right] = \frac{1}{2}B - \frac{1}{2}B\int_0^1 d\rho\,\rho^{-1} \tag{D.29}$$

Substituting (D.24) through (D.29) into (D.22)

$$\begin{aligned} J(A,B,C) = \frac{(4-2\varepsilon)\Gamma(2\varepsilon-3)}{4\,(4\pi)^{4-2\varepsilon}}\Bigg[&\frac{(3-2\varepsilon)}{2\varepsilon}\left(B^{1-2\varepsilon}+C^{1-2\varepsilon}\right)+\frac{(6-2\varepsilon)(3-2\varepsilon)(2-2\varepsilon)}{6\varepsilon\,A^{2\varepsilon-1}} \\ &+2A+\frac{9}{2}(B+C)+\varepsilon H(A,B,C)+O(\varepsilon^2)\Bigg] \end{aligned} \tag{D.30}$$

Now $\Gamma(2\varepsilon-3)=\Gamma(2\varepsilon-1)/(2\varepsilon-3)(2\varepsilon-2)$. We can thus perform a Taylor expansion in ε on all but $\Gamma(2\varepsilon-1)$ and expand $\Gamma(2\varepsilon-1)$ using (B.5) to obtain

$$\begin{aligned} J(A,B,C) = \frac{-1}{12\,(16\pi^2)^2}\Bigg[&\frac{3(4A+B+C)}{2\varepsilon^2}+\frac{1}{\varepsilon}\Big\{3\psi(2)(4A+B+C) \\ &-3A+\frac{21}{4}(B+C)-12A\ln(A/4\pi)-3B\ln(B/4\pi) \\ &-3C\ln(C/4\pi)\Big\}\Bigg]+F_J(A,B,C)+O(\varepsilon) \end{aligned} \tag{D.31}$$

where

$$
\begin{aligned}
F_J(A,B,C) = -\tfrac{1}{12}(16\pi^2)^{-2}\Big[& H(A,B,C) + 12A\ln^2 A + 3B\ln^2 B \\
& + 3C\ln^2 C + 24A\ln A + 2B\ln B + 2C\ln C + \ln^2 4\pi\,(12A+3B+3C) \\
& + 2\ln 4\pi\left(-12A\ln A - 3B\ln B - 3C\ln C - 10A + \tfrac{7}{2}B + \tfrac{7}{2}C\right) \\
& + \psi(2)\left(-24A\ln(A/4\pi) - 6B\ln(B/4\pi) - 6C\ln(C/4\pi) - 6A + \tfrac{21}{2}B + \tfrac{21}{2}C\right) \\
& - 14A\ln(A/4\pi) - \tfrac{7}{2}B\ln(B/4\pi) - \tfrac{7}{2}C\ln(C/4\pi) \\
& + (12A+3B+3C)\left(\tfrac{\pi^2}{3} + \psi^2(2) - \psi'(2)\right) + \tfrac{10}{3}A + 6B + 6C\Big]
\end{aligned}
\tag{D.32}
$$

APPENDIX E - EVALUATION OF K(A,B,C)

In this appendix we evaluate the integral

$$(E.1)\quad K(A,B,C)=\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\frac{(k_\mu-p_\mu)(k_\nu-p_\nu)\left[-g_{\mu\nu}+(k_\mu+p_\mu)(k_\nu+p_\nu)/(k+p)^2\right]}{((k+p)^2+A)(k^2+B)(p^2+C)}$$

where the Euclidean metric ($g_{\mu\nu}=\delta_{\mu\nu}$) has been used. Equation (E.1) can be written in the form

$$(E.2)\quad K(A,B,C)=\int \frac{d^n p}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\frac{\left[-(k-p)^2+(k^2-p^2)^2/(k+p)^2\right]}{((k+p)^2+A)(k^2+B)(p^2+C)}$$

Consider the numerator of the integrand

$$(E.3)\quad K_0 = -(k-p)^2+(k^2-p^2)^2(k+p)^{-2}$$

One can expand and insert $0=(k^2-p^2)(C-B)+Bk^2-Bp^2-Ck^2+Cp^2$ to obtain

$$(E.4)\quad K_0=-k^2+2k\cdot p-p^2+\frac{\left[k^4-2k^2p^2+p^4+(k^2-p^2)(C-B)+k^2B-p^2B-k^2C+p^2C\right]}{(k+p)^2}$$

Inserting $0=2(k^2+p^2)-2(k^2+p^2)$ and regrouping terms gives

$$(E.5)\quad K_0=(k+p)^2-2(k^2+p^2)+\frac{\left[(C-B)(k^2-p^2)+k^2(k^2+B)+p^2(p^2+C)-p^2(k^2+B)-k^2(p^2+C)\right]}{(k+p)^2}$$

One can combine appropriate cancelling factors of A,B,C to write (E.5) in the form

$$K_0 = \left((k+p)^2+A\right) - A - 2(k^2+B) - 2(p^2+c) + 2B + 2C + \frac{(C-B)^2}{A}\cdot\frac{A}{(k+p)^2}$$

$$+\frac{(C-B)}{A}\frac{A}{(k+p)^2}\left[(k^2+B)-(p^2+C)\right] + \frac{1}{A}\frac{A}{(k+p)^2}\Big[k^2(k^2+B)+p^2(p^2+C) \tag{E.6}$$

$$-p^2(k^2+B) - k^2(p^2+C)\Big]$$

If we add $0=(k+p)^2-(k+p)^2$ to A in the numerators, K_0 becomes

$$K_0 = \left((k+p)^2+A\right) - 2(k^2+B) - 2(p^2+c) + (2B+2C-A) + A^{-1}(C-B)\Big[-(k^2+B)$$

$$+(k^2+B)\left((k+p)^2+A\right)(k+p)^{-2} + (p^2+C) - (p^2+c)\left((k+p)^2+A\right)(k+p)^{-2}\Big]$$

$$+A^{-1}(C-B)^2\left[\left((k+p)^2+A\right)(k+p)^{-2}-1\right] + A^{-1}\Big[k^2(k^2+B)\left((k+p)^2+A\right)(k+p)^{-2}$$

$$-k^2(k^2+B) + p^2(p^2+c)\left((k+p)^2+A\right)(k+p)^{-2} - p^2(p^2+C) + p^2(k^2+B) \tag{E.7}$$

$$-p^2(k^2+B)\left((k+p)^2+A\right)(k+p)^{-2} + k^2(p^2+c)$$

$$-k^2(p^2+c)\left((k+p)^2+A\right)(k+p)^{-2}\Big]$$

Substituting (E.7) into (E.2)

$$K(A,B,C) = K_1(B,C) - 2K_2(A,C) - 2K_2(A,B) + (2B+2C-A)\,I(A,B,C)$$

$$-A^{-1}(C-B)\left[K_2(A,C) - K_2(0,C) - K_2(A,B) + K_2(0,B)\right]$$

$$-A^{-1}(C-B)^2\left[I(A,B,C) - I(0,B,C)\right] \tag{E.8}$$

$$+A^{-1}\Big[K_3(0,C) - K_3(A,C) + K_3(0,B) - K_3(A,B) - K_4(0,c)$$

$$+K_4(A,C) - K_4(0,B) + K_4(A,B)\Big]$$

where

$$K_1(A,B)=\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}(k^2+A)^{-1}(p^2+B)^{-1} \tag{E.9}$$

$$K_2(A,B)=\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}((k+p)^2+A)^{-1}(p^2+B)^{-1} \tag{E.10}$$

$$K_3(A,B)=\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}p^2((k+p)^2+A)^{-1}(p^2+B)^{-1} \tag{E.11}$$

$$K_4(A,B)=\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}k^2((k+p)^2+A)^{-1}(p^2+B)^{-1} \tag{E.12}$$

$$I(A,B,C)=\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}((k+p)^2+A)^{-1}(k^2+B)^{-1}(p^2+C)^{-1} \tag{E.13}$$

With the variable change $k'=p+k, p'=-k$ the expressions for K_2, K_3 and K_4 become

$$K_2(A,B)=K_1(A,B) \tag{E.14}$$

$$K_3(A,B)=-(A+B)K_1(A,B)+\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}\left[(k^2+A)^{-1}+(p^2+B)^{-1}\right] \tag{E.15}$$

$$K_4(A,B)=-BK_1(A,B)+\int\frac{d^np}{(2\pi)^n}\int\frac{d^nk}{(2\pi)^n}(k^2+A)^{-1} \tag{E.16}$$

Substituting (E.14),(E.15),(E.16) into (E.8)

$$\begin{aligned}K(A,B,C) &= K_1(B,C)-K_1(A,C)-K_1(A,B)+(2B+2C-A)I(A,B,C)\\ &+A^{-1}(B-C)\left[K_1(A,C)-K_1(0,C)-K_1(A,B)+K_1(0,B)\right]\\ &-A^{-1}(B-C)^2\left[I(A,B,C)-I(0,B,C)\right]\end{aligned} \tag{E.17}$$

Applying (B.1) to (E.9) and then setting $n=4-2\varepsilon$ gives

$$K_1(A,B) = (4\pi)^{2\varepsilon-4}\,\Gamma^2(\varepsilon-1)\,(AB)^{1-\varepsilon} \tag{E.18}$$

Substituting (E.18) into (E.17)

$$\begin{aligned} K(A,B,C) &= (2B+2C-A)\,I(A,B,C) - A^{-1}(B-C)^2\big(I(A,B,C)-I(0,B,C)\big) \\ &+ \frac{\Gamma^2(\varepsilon-1)}{(4\pi)^{4-2\varepsilon}}\Big\{(BC)^{1-\varepsilon}-(AC)^{1-\varepsilon}-(AB)^{1-\varepsilon}+A^{-\varepsilon}\big[BC^{1-\varepsilon}-C^{2-\varepsilon}-B^{2-\varepsilon}+CB^{1-\varepsilon}\big]\Big\} \end{aligned} \tag{E.19}$$

Expanding all but $\Gamma(\varepsilon-1)$ in a Taylor series in ε yields

$$\begin{aligned} K(A,B,C) &= (2B+2C-A)\,I(A,B,C) - A^{-1}(B-C)^2\big(I(A,B,C)-I(0,B,C)\big) \\ &+ \Gamma^2(\varepsilon-1)(16\pi^2)^{-2}\Big\{(3BC-AC-AB-B^2-C^2) + \varepsilon\big[(B+A-2C)B\ln(B/4\pi) \\ &+ (AC+AB+B^2+C^2-2BC)\ln(A/4\pi) + (C+A-2B)C\ln(C/4\pi)\big] \\ &+ \varepsilon^2 L(A,B,C) + O(\varepsilon^3)\Big\} \end{aligned} \tag{E.20}$$

where

$$\begin{aligned} 2L(A,B,C) &= BC\ln^2 BC - AC\ln^2 AC - AB\ln^2 AB + BC\ln^2 C \\ &- C^2\ln^2 C - B^2\ln^2 B + BC\ln^2 B - (B-C)^2\ln^2 A \\ &- 2\ln A\,(-BC\ln C + C^2\ln C + B^2\ln B - BC\ln B) \\ &+ 4\ln 4\pi\big[\ln A\,(AC+AB+B^2+C^2-2BC) \\ &+ B(B+A-2C)\ln B + C(C+A-2B)\ln C\big] \\ &+ 4\ln^2 4\pi\,(3BC-AC-AB-B^2-C^2) \end{aligned} \tag{E.21}$$

One can expand $\Gamma(\varepsilon-1)$ with (B.5) to obtain

$$K(A,B,C)=(2B+2C-A)\left(I(A,B,C)-F_I(A,B,C)\right)-A^{-1}(B-C)^2\left[I(A,B,C)\right.$$

$$\left.-F_I(A,B,C)-I(0,B,C)+F_I(0,B,C)\right]+(16\pi^2)^{-2}\Big\{\varepsilon^{-2}(3BC-AC-AB-B^2-C^2)$$

$$+\varepsilon^{-1}\Big[2\psi(2)(3BC-AC-AB-B^2-C^2)+B(B+A-2C)\ln(B/4\pi)+(AC+AB \tag{E.22}$$

$$\left.\left.+B^2+C^2-2BC\right)\ln(A/4\pi)+C(C+A-2B)\ln(C/4\pi)\Big]\Big\}+F_K(A,B,C)+O(\varepsilon)$$

where

$$F_K(A,B,C)=(2B+2C-A)F_I(A,B,C)-A^{-1}(B-C)^2\left[F_I(A,B,C)-F_I(0,B,C)\right]$$

$$+(16\pi^2)^{-2}\Big\{L(A,B,C)+2\psi(2)\Big[(AC+AB+B^2+C^2-2BC)\ln(A/4\pi) \tag{E.23}$$

$$+B(B+A-2C)\ln(B/4\pi)+C(C+A-2B)\ln(C/4\pi)\Big]$$

$$+\psi^2(2)(3BC-AC-AB-B^2-C^2)\Big\}$$

and $F_I(A,B,C)$ is given by (C.25). Finally one substitutes (C.24) into (E.22) with the result

$$K(A,B,C)=(16\pi^2)\Big\{\tfrac{1}{2}\varepsilon^{-2}\left(A^2-3A(B+C)-3(B^2+C^2)\right)$$

$$+\varepsilon^{-1}\Big[\psi(2)\left(A^2-3A(B+C)-3(B^2+C^2)\right)+\tfrac{1}{2}\left(A^2-A(B+C)-B^2-C^2-6BC\right) \tag{E.24}$$

$$-A(A-3B-3C)\ln(A/4\pi)+3B^2\ln(B/4\pi)$$

$$+3C^2\ln(C/4\pi)\Big]\Big\}+F_K(A,B,C)+O(\varepsilon)$$

BIOGRAPHICAL INFORMATION

NAME:

Ross Taylor Bates

MAILING ADDRESS:

Apt. #704 - 4660 West Tenth Avenue
Vancouver, British Columbia. V6R 2J6

PLACE AND DATE OF BIRTH:

London, Ontario September 4, 1957.

EDUCATION (Colleges and Universities attended, dates, and degrees):

University of Western Ontario (1976-1980)
B.Sc. (Hons.) in Applied Mathematics (1980)

University of British Columbia (1980-present)

POSITIONS HELD:

Research Assistant at U.W.O. (May-August, 1979)

Research Assistant at U.W.O. (May-June, 1980)

PUBLICATIONS (if necessary, use a second sheet):

AWARDS:

NSERC 1967 Science Scholarship (1980-present)

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